Gaussian Solitary Waves in Granular Chains

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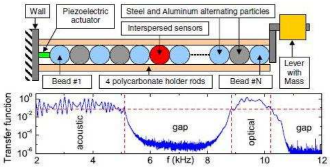
Introduction



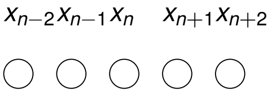
- Granular chains contain densely packed, elastically interacting particles with Hertzian contact forces.
- Recent works focus on solitary and periodic traveling waves in granular chains; as they are relevant to physical experiments.

On solitary travelling waves in granular chains

Existence of solitary waves was proved with the variational theory based on the differential–difference equation.

- G. Friesecke and J. Wattis, *Commun. Math. Phys.* **161** (1994), 391 - proof of existence for a general FPU lattice
- R. MacKay, *Phys. Lett. A* **251** (1999), 191 - adaptation of this method to granular chains
- J. English and R. Pego, *Proc. Amer. Math. Soc.* **133** (2005), 1763 - proof of the double-exponential tails of the solitary waves
- A. Stefanov and P. Kevrekidis, *J. Nonlinear Sci.* **22** (2012), 327 - proof of the bell-shaped profile of the solitary waves

The granular chain



Newton's equations define the FPU (Fermi-Pasta-Ulam) lattice:

$$\frac{d^2 x_n}{dt^2} = V'(x_{n+1} - x_n) - V'(x_n - x_{n-1}), \quad n \in \mathbb{Z},$$

where x_n is the displacement of the nth particle.

The interaction potential for spherical beads is

$$V(x) = \frac{1}{1+\alpha}|x|^{1+\alpha} H(-x), \quad \alpha = \frac{3}{2},$$

where H is the step (Heaviside) function.

H. Hertz, *J. Reine Angewandte Mathematik* **92** (1882), 156

Travelling waves and the Boussinesq approximation

Using the relative displacements $u_n = x_n - x_{n-1}$ and applying the travelling wave reduction $u_n(t) = w_n(n-t)$, we obtain

$$\frac{d^2 w}{dz^2} = \Delta(w|w|^{\alpha-1}), \quad z \in \mathbb{R},$$

with $(\Delta w)(z) = w(z+1) - 2w(z) + w(z-1)$.

Expanding $\Delta = \partial_z^2 + \frac{1}{12}\partial_z^4$ and integrating twice, we obtain

$$w = w|w|^{\alpha-1} + \frac{1}{12}\frac{d^2}{dz^2} w|w|^{\alpha-1}, \quad z \in \mathbb{R},$$

which has compactons

$$w_{\mathrm{c}}(z) = \begin{cases} A\cos^{\frac{2}{\alpha-1}}(Bz), & |z| \leq \frac{\pi}{2B}, \\ 0, & |z| \geq \frac{\pi}{2B}, \end{cases}$$

where

$$A = \left(\frac{1+\alpha}{2\alpha}\right)^{\frac{1}{1-\alpha}}, \quad B = \frac{\sqrt{3}(\alpha-1)}{\alpha}.$$

Ill-posedness of the Boussinesq equation

The fully nonlinear Boussinesq equation takes the form

$$u_{tt} = (u|u|^{\alpha-1})_{xx} + \frac{1}{12}(u|u|^{\alpha-1})_{xxxx},$$

V.F. Nesterenko, *J. Appl. Mech. Tech. Phys.* **24** (1983), 733
K. Ahnert and A. Pikovsky, *Phys. Rev. E* **79** (2009), 026209.

Cauchy problem for the Boussinesq equation is ill-posed.

Compare with the recent work on ill-posedness of degenerate dispersive equations:

D.M. Ambrose, G. Simpson, J.D. Wright, and D.G. Yang, *Nonlinearity* **25** (2012), 2655.

Korteweg–de Vries equation in the case of precompression

Consider again the FPU lattice

$$\frac{d^2 u_n}{dt^2} = V'(u_{n+1}) - 2V'(u_n) + V'(u_{n-1}), \quad n \in \mathbb{Z}.$$

If $V \in C^3$ with $V''(0) = \kappa > 0$ and $V'''(0) \neq 0$, then the asymptotic multi-scale expansion

$$u_n(t) = \kappa (4V'''(0))^{-1} \varepsilon^2 y(\xi, \tau) + \text{higher order terms},$$

where $\xi := \varepsilon(n - c_s t)$, $\tau := \varepsilon^3 c_s t/24$, and $c_s := \sqrt{\kappa}$ is the "sound velocity" of linear waves, shows that y satisfies the KdV equation

$$\partial_\tau y + 3y\partial_\xi y + \partial_\xi^3 y = 0.$$

The KdV equation admits the solitary waves $y = \mathrm{sech}^2((\xi - \tau)/2)$.

Relevant results

- The KdV equation can be justified at a time scale of order ε^{-3}.
  G. Schneider and C.E. Wayne, *International Conference on Differential Equations Appl.* **5** (1998) 69
  D. Bambusi, A. Ponno, *Comm. Math. Phys.* **264** (2006), 539

- Nonlinear stability of small amplitude FPU solitons can be proved.
  G. Friesecke and R.L. Pego, *Nonlinearity* **12** (1999), 1601; **15** (2002), 1343; **17** (2004), 207; **17** (2004), 229.

- Existence and stability of *N*-soliton solutions can be proved.
  A. Hoffman and C.E. Wayne, *Nonlinearity* **21** (2008), 2911; *J. Dyn. Diff. Equat.* **21** (2009), 343.
  T. Mizumachi, *Commun. Math. Phys.* **288** (2009), 125; *SIMA* **43** (2011), 2170; *Arch. Rat. Mech. Anal.* **207** (2013), 393.

Korteweg–de Vries equation without precompression

Consider again the FPU lattice

$$\left(\frac{d^2}{dt^2}-\Delta\right)u_n=\Delta f_\alpha(u_n),\quad n\in\mathbb{Z},$$

where

$$f_\alpha(u):=u(|u|^{\alpha-1}-1)=(\alpha-1)\,u\ln|u|+\mathcal{O}((\alpha-1)^2),$$

as $\alpha\to1$. For the chains of hollow spherical particles of different width, α is defined in the range $1.1\le\alpha\le1.5$.

Let $\alpha=1+\varepsilon^2$. Using the asymptotic multi-scale expansion

$$u_n(t)=v(\xi,\tau)+\text{higher order terms},$$

where $\xi:=2\sqrt{3}\varepsilon(n-t)$, $\tau:=\sqrt{3}\varepsilon^3 t$, we obtain the KdV equation with the logarithmic nonlinearity (log-KdV)

$$\partial_\tau v+\partial_\xi(v\log v)+\partial_\xi^3 v=0.$$

Stationary solutions

Stationary log-KdV equation can be integrated once to get

$$\frac{d^2 v}{d\xi^2} + v \ln|v| = 0,$$

which admits the Gaussian solitons

$$v(\xi) = \sqrt{e}\, e^{-\xi^2/4}.$$

A. Chatterjee, PRE **59** (1999), 5912

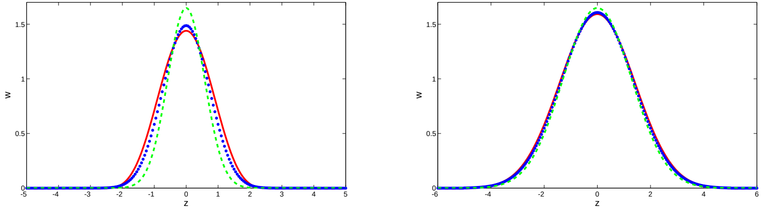


Figure : Solitary waves (blue) in comparison with the compactons (red) and the Gaussian solitons (green) for $\alpha = 1.5$ (left) and $\alpha = 1.1$ (right).

Numerical evidence of convergence of the approximation

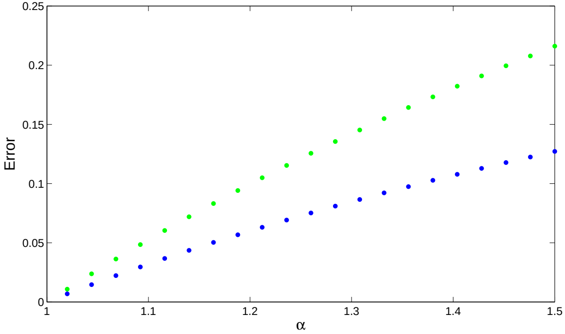


Figure : The L^∞ distance between solitary waves of the FPU chain and either Nesterenko compactons (blue dots) or Gaussian solitons (green dots) vs. α.

Numerical evidence of stability

Lattice of $N = 2000$ particles is excited with the initial impact

$$\dot{x}_n(0) = 0.1\delta_{n,0}, \quad \dot{x}_n(0) = 0 \text{ for all } n \geq 1.$$

A Gaussian solitary wave is formed asymptotically as t evolves.

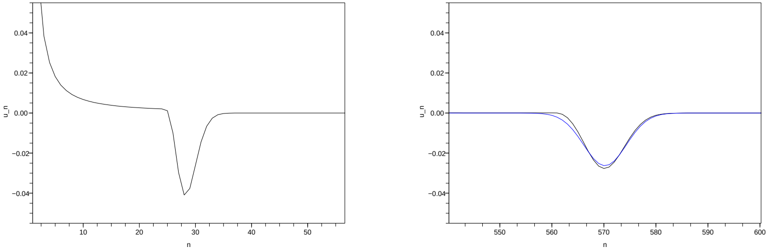


Figure : Formation of a Gaussian wave (blue curve) in the Hertzian FPU lattice with $\alpha = 1.01$: $t \approx 30.5$ (left) and $t \approx 585.6$ (right).

Summary of main results

The log-KdV equation

$$\partial_\tau v + \partial_\xi (v \log v) + \partial_\xi^3 v = 0.$$

1. Gaussian solitary wave is linearly orbitally stable in space $H^1(\mathbb{R})$.
2. For any initial data v_0 from the energy space $X \subset H^1(\mathbb{R})$, there exists a global solution $v \in L^\infty(\mathbb{R}, X)$ s.t. the energy is not increasing in time. Uniqueness is not proved.
3. The spectrum of the linearized operator in $L^2(\mathbb{R})$ is purely discrete and consists of a double zero eigenvalue and a symmetric sequence of simple purely imaginary eigenvalues $\{\pm i\omega_n\}_{n\in\mathbb{N}}$ s.t. $0 < \omega_1 < \omega_2 < \ldots$ and $\omega_n \to \infty$ as $n \to \infty$. The eigenfunctions decay algebraically as $|\xi| \to \infty$.
4. The log–KdV equation is justified as a valid approximation for the FPU lattices for solutions v bounded away from zero.

Energy functional

The log-KdV equation

$$\partial_\tau v + \partial_\xi (v \log v) + \partial_\xi^3 v = 0.$$

can be written in the Hamiltonian form

$$\partial_\tau v = \partial_\xi E'(v),$$

where the energy functional is

$$E(v) = \frac{1}{2} \int_{\mathbb{R}} \left[(\partial_\xi v)^2 - v^2 \left(\log v - \frac{1}{2} \right) \right] d\xi.$$

Gaussian wave $v_0 = e^{\frac{2-\xi^2}{4}}$ is a critical point of $E(v)$: $E'(v_0) = 0$, and

$$L = E''(v_0) = -\partial_\xi^2 - 1 - \log(v_0) = -\frac{\partial^2}{\partial \xi^2} - \frac{3}{2} + \frac{\xi^2}{4}.$$

Linear operators and evolution

Theorem 1 (G.James, D.P., 2014)

v_0 is linearly orbitally stable in space $H^1(\mathbb{R})$.

Consider the time evolution of the perturbation u to v_0:

$$\partial_\tau u = \partial_\xi L u, \quad u(0) = u_0.$$

The solitary wave is linearly orbitally stable if for every $u_0 \in \mathrm{Dom}(\partial_x L)$ such that $\langle v_0, u_0 \rangle_{L^2} = 0$ there exists constant $C(u_0)$ such that

$$\|u(\tau)\|_{H^1} \leq C(u_0), \quad \tau \in \mathbb{R}.$$

The operator L is self-adjoint in $L^2(\mathbb{R})$ with domain

$$\mathrm{Dom}(L) = \{ u \in H^2(\mathbb{R}), \xi^2 u \in L^2(\mathbb{R}) \}.$$

The spectrum of L consists of simple eigenvalues at $n-1$, $n \in \mathbb{N}_0$.

Symplectic decomposition

We know that $\partial_\xi L$ has a double zero eigenvalue because

$$L v_0' = 0, \quad \partial_\xi L\, v_0 = -v_0',$$

and no $u \in \mathrm{Dom}(\partial_\xi L)$ exists in $\partial_\xi L u = v_0$ because $\|v_0\|_2^2 \neq 0$.

Using the decomposition

$$u(\xi,\tau) = a(\tau)\, v_0'(\xi) + b(\tau)\, v_0(\xi) + y(\xi,\tau)$$

with $\langle v_0, y\rangle_{L^2} = 0$ and $\langle \partial_\xi^{-1} v_0, y\rangle_{L^2} = 0$, we obtain

$$\frac{da}{d\tau} + b = 0, \quad \frac{db}{d\tau} = 0, \quad \frac{\partial y}{\partial \tau} = \partial_\xi L\, y.$$

If $\langle v_0, u_0\rangle_{L^2} = 0$, then $b(\tau) = b(0) = 0$ and $a(\tau) = a(0)$.

Proof of linear orbital stability

Because v_0 and v_0' are eigenvectors of L for the negative and zero eigenvalues, L is strictly positive definite on ${v_0}^{\perp} \cap {v_0'}^{\perp} \subset L^2(\mathbb{R})$.

As a result, $\|y\|_L = \langle Ly, y \rangle_{L^2}^{1/2}$ defines a norm (equivalent to a weighted H^1-norm).

From the energy balance,

$$\frac{d}{d\tau} \frac{1}{2} \|y\|_L^2 = \langle Ly, \partial_\tau y \rangle_{L^2} = \langle Ly, \partial_\xi L y \rangle_{L^2} = 0,$$

we obtain the Lyapunov stability of the zero equilibrium $y = 0$ in the constrained space $\langle v_0, y \rangle_{L^2} = 0$ and $\langle \partial_\xi^{-1} v_0, y \rangle_{L^2} = 0$.

The constrained space corresponds to the modulation of the two parameters of the Gaussian solitary wave.

Global existence of solutions

The log-KdV equation

$$\partial_\tau v + \partial_\xi (v \log v) + \partial_\xi^3 v = 0$$

has the associated energy functional

$$E(v) = \frac{1}{2} \int_{\mathbb{R}} \left[(\partial_\xi v)^2 - v^2 \left(\log v - \frac{1}{2} \right) \right] d\xi,$$

defined in the function space

$$X := \left\{ v \in H^1(\mathbb{R}) : \quad v^2 \log|v| \in L^1(\mathbb{R}) \right\}.$$

Theorem 2 (R. Carles, D.P., 2014)

For any $v_0 \in X$, there exists a global solution $v \in L^\infty(\mathbb{R}, X)$ of the log–KdV equation such that

$$\|v(\tau)\|_{L^2} \leq \|v_0\|_{L^2}, \quad E(v(\tau)) \leq E(v_0), \quad \text{for all } \tau \in \mathbb{R}.$$

Proof of global existence

1. Construct an approximation of the logarithmic nonlinearity (Cazenave, 1980):

$$f_\varepsilon(v) = \begin{cases} v\log(v), & |v| \geq \varepsilon, \\ \left(\log(\varepsilon) - \frac{3}{4}\right)v + \frac{1}{\varepsilon^2}v^3 - \frac{1}{4\varepsilon^4}v^5, & |v| \leq \varepsilon, \end{cases}$$

hence $f_\varepsilon \in C^2(\mathbb{R})$ and $f_\varepsilon(v) \to v\log(v)$ as $\varepsilon \to 0$ for every $v \in \mathbb{R}$.

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2. Obtain existence of the global approximating solutions $v^\varepsilon \in C(\mathbb{R}, H^1(\mathbb{R}))$ of the generalized KdV equations

$$\begin{cases} v^\varepsilon_\tau + v^\varepsilon_{\xi\xi\xi} + f'_\varepsilon(v^\varepsilon) v^\varepsilon_\xi = 0, & \tau > 0, \\ v^\varepsilon|_{\tau=0} = v_0. \end{cases}$$

   (Kenig, Ponce, Vega, 1991).

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3. Obtain uniform estimates for all $\varepsilon > 0$ and $\tau \in \mathbb{R}$:

$$\|v^\varepsilon(\tau)\|_{H^1} + \|(v^\varepsilon(\tau))^2 \log(v^\varepsilon(\tau))\|_{L^1} \leq C(v_0).$$

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4. Pass to the limit $\varepsilon \to 0$ and obtain a global solution $v \in L^\infty(\mathbb{R}, X)$ of the log–KdV equation.

Uniqueness

Lemma: *Assume that a solution $v \in L^\infty(\mathbb{R}, X)$ of the log–KdV equation satisfies the additional condition*

$$(\log|v|)_\xi \in L^\infty([-\tau_0, \tau_0] \times \mathbb{R}).$$

Then, the solution v is unique for every $\tau \in (-\tau_0, \tau_0)$, depends continuously on the initial data $v_0 \in X$, and satisfies $\|v(\tau)\|_{L^2} = \|v_0\|_{L^2}$ and $E(v(\tau)) = E(v_0)$ for all $\tau \in (-\tau_0, \tau_0)$.

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- $\partial_\xi \log|v|$ is unbounded as $|\xi| \to \infty$ for the Gaussian solitary wave.
- Nonlinear orbital stability of Gaussian solitary wave is conditional that the global solution $v \in L^\infty(\mathbb{R}, X)$ is unique and depends continuously on the initial data $v_0 \in X$.

Spectral stability

If $v = V(\xi)e^{\lambda\tau}$, we arrive to the linear eigenvalue problem

$$\partial_\xi L V = \lambda V.$$

where we recall that $\sigma(L) = \{n-1, \quad n \in \mathbb{N}_0\}$ and the eigenfunctions of L have Gaussian decay in ξ.

Theorem 3 (R. Carles, D.P., 2014)

The spectrum of $\partial_x L$ in $L^2(\mathbb{R})$ is purely discrete and consists of a double zero eigenvalue and a symmetric sequence of simple purely imaginary eigenvalues $\{\pm i\omega_n\}_{n\in\mathbb{N}}$ such that $0 < \omega_1 < \omega_2 < \dots$ and $\omega_n \to \infty$ as $n \to \infty$. The eigenfunctions for nonzero eigenvalues are smooth in ξ but decay algebraically as $|\xi| \to \infty$.

Further remarks

- Because the spectrum of $\partial_x L$ is purely discrete, no asymptotic stability result can hold for Gaussian solitary waves.
- This agrees with the result of Cazenave for the log–NLS equation: the L^p norms at the solution v for any $p \geq 2$ including $p = \infty$ may not vanish as $t \to \infty$ (or in a finite time).

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- Nonlinear analysis of perturbations to the Gaussian solitary wave becomes now problematic. If $v(\xi,\tau) := v_0(\xi) + w(\xi,\tau)$ is set, then w satisfies

$$w_\tau = \partial_\xi L w - \partial_\xi N(w),$$

  where

$$N(w) := w \log\left(1 + \frac{w}{v_0}\right) + v_0 \left[\log\left(1 + \frac{w}{v_0}\right) - \frac{w}{v_0}\right].$$

  However, w/v_0 may grow like an inverse Gaussian function of ξ.

Proof of spectral stability

The linear eigenvalue problem

$$AV = \lambda V, \quad A := \partial_\xi L = -\partial_\xi^3 + \frac{1}{4}(\xi^2 - 6)\partial_\xi + \frac{1}{2}\xi,$$

can be written in the equivalent form with the Fourier transform

$$\hat{A}\hat{V} = \lambda \hat{V}, \quad \hat{A} = \frac{i}{4}k\left(-\partial_k^2 + 4k^2 - 6\right).$$

with the natural choice $\lambda = \frac{i}{4}E$.

Eigenfunctions of A are defined in the domain $X_A := D(A) \cap \dot{H}^{-1}(\mathbb{R})$,

$$D(A) = \left\{u \in H^3(\mathbb{R}) : \quad \xi^2 \partial_\xi u \in L^2(\mathbb{R}), \quad \xi u \in L^2(\mathbb{R})\right\}.$$

In the Fourier form, the domain X_A becomes

$$\hat{X}_A = \left\{\hat{u} \in H^1(\mathbb{R}) : \quad k\partial_k^2 \hat{u} \in L^2(\mathbb{R}), \quad k^3 \hat{u} \in L^2(\mathbb{R}), \quad k^{-1}\hat{u} \in L^2(\mathbb{R})\right\}.$$

Proof of spectral stability

The linear eigenvalue problem is

$$\frac{d^2\hat{u}}{dk^2} + \left(\frac{E}{k} + 6 - 4k^2\right)\hat{u}(k) = 0, \quad k \in \mathbb{R}.$$

- As $k \to 0$, two linearly independent solutions exist

  $$\hat{u}_1(k) = k + \mathcal{O}(k^2), \quad \hat{u}_2(k) = 1 + \mathcal{O}(k\log(k)).$$

  The second solution does not belong to $\hat{X}_A$.

- As $|k| \to \infty$, the decaying solution satisfies

  $$\hat{u}(k) = ke^{-k^2}\left(1 + \mathcal{O}(|k|^{-1})\right).$$

  The shooting problem is over-determined.

Proof of spectral stability

- The way around is the weak piecewise definition of the eigenfunction:

$$\hat{u}(k) = \begin{cases} \hat{u}_+(k), & k > 0, \\ 0, & k < 0, \end{cases} \quad \text{or} \quad \hat{u}(k) = \begin{cases} 0, & k > 0, \\ \hat{u}_-(k), & k < 0, \end{cases}$$

where $\hat{u}_\pm(0) = 0$, so that $\hat{u} \in \hat{X}_A$.

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  where $\hat{u}_\pm(0) = 0$, so that $\hat{u} \in \hat{X}_A$.

- For $\hat{u}_+$, we set $\hat{u}_+(k) = k^{1/2}\hat{v}_+(k)$ and obtain

$$k^{1/2}\left(-\frac{d^2}{dk^2} + 4k^2 - 6\right)k^{1/2}\hat{v}_+(k) = E\hat{v}_+(k), \quad k \in (0,\infty),$$

  which is now in the symmetric form. Hence $E \in \mathbb{R}$.

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which is now in the symmetric form. Hence $E \in \mathbb{R}$.

- For $E = 0$, we have $\hat{v}_+ = k^{1/2}e^{-k^2} > 0$ for $k > 0$. By Sturm's Theorem, the set of eigenvalues $\{E_n\}_{n\in\mathbb{N}_0}$ satisfies $0 = E_0 < E_1 < E_2 < \ldots$ and $E_n \to \infty$ as $n \to \infty$.

Numerical illustration

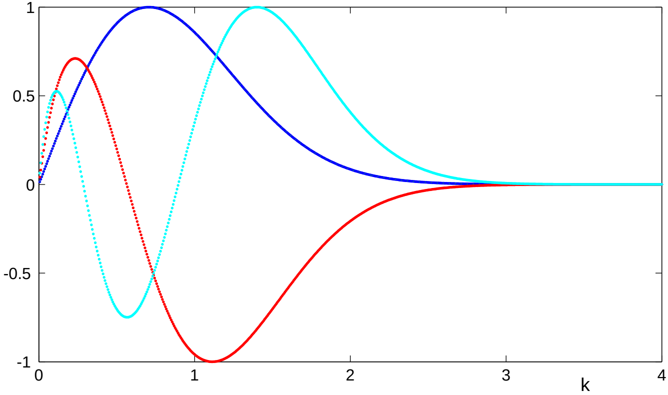

Figure : Eigenfunctions $\hat{u}$ of the spectral problem versus k for the first three eigenvalues $E_0 = 0$, $E_1 \approx 5.411$, and $E_2 \approx 12.308$.

Precompression

Consider again the FPU lattice in the form

$$\frac{d^2 u_n}{dt^2} = -(\Delta\, |u|^{1+\varepsilon^2} H(-u))_n, \quad n \in \mathbb{Z}.$$

Let $u_n(t) = -1 - w_n(t)$, then $w_n(t)$ satisfies

$$\frac{d^2 w_n}{dt^2} = (\Delta\, V_\varepsilon(w))_n, \quad n \in \mathbb{Z}$$

with the regularized potential

$$V_\varepsilon(w) := \frac{1}{2+\varepsilon^2}\left[(1+w)^{2+\varepsilon^2} - 1\right] - w, \quad w > -1$$

Precompression

Consider again the FPU lattice in the form

$$\frac{d^2 u_n}{dt^2} = -(\Delta |u|^{1+\varepsilon^2} H(-u))_n, \quad n \in \mathbb{Z}.$$

Let $u_n(t) = -1 - w_n(t)$, then $w_n(t)$ satisfies

$$\frac{d^2 w_n}{dt^2} = (\Delta V_\varepsilon(w))_n, \quad n \in \mathbb{Z}$$

with the regularized potential

$$V_\varepsilon(w) := \frac{1}{2+\varepsilon^2}\left[(1+w)^{2+\varepsilon^2} - 1\right] - w, \quad w > -1$$

The KdV scaling $w_n(t) \approx W(\xi, \tau)$ with $\xi := \varepsilon(n-t)$ and $\tau := \varepsilon^3 t$ yields the log–KdV equation

$$2\partial_\tau W + \frac{1}{12}\partial_\xi^3 W + \partial_\xi((1+W)\log(1+W)) = 0.$$

Traveling waves of the log–KdV equation

Traveling waves $W(\xi - \lambda\tau/2)$ satisfy the stationary log-KdV equation

$$\lambda W(x) = \frac{1}{12} W''(x) + (1+W)\log(1+W), \quad x \in \mathbb{R},$$

where $(1+W)\log(1+W) = W + W^2/2 + \mathcal{O}(W^3)$.

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For any $\lambda > 1$, there exists a unique even solution $W \in H^1(\mathbb{R})$ of the stationary log–KdV equation. Moreover,

- $W(x) > 0$ for all $x \in \mathbb{R}$,
- $W(x) \to 0$ as $|x| \to \infty$ exponentially fast,
- $W \in H^\infty(\mathbb{R})$,
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The travelling solitary wave is orbitally stable in the log–KdV equation
J. Höwing, J. Diff. Eqs. **251** (2011), 2515.

Main results

Theorem 4 (E.Dumas–D.P., 2014)

For every $\lambda > 1$, there exist positive constants ε_0 and C_0 s.t. for every $\varepsilon \in (0, \varepsilon_0)$, there exists a unique even travelling solution $w_{\mathrm{stat},\varepsilon}$ of the FPU lattice in $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ s.t.

$$\sup_{z\in\mathbb{R}} |w_{\mathrm{stat},\varepsilon}(z) - W_{\mathrm{stat}}(\varepsilon z)| \leq C_0 \varepsilon^{1/6},$$

where W_{stat} is the unique even solution to the log–KdV equation.

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where $W_{\rm stat}$ is the unique even solution to the log–KdV equation.

Remarks:

- Moreover, $w_{\rm stat,\varepsilon} \in H^\infty(\mathbb{R})$.
- Moreover, $w_{\rm stat,\varepsilon}$ decays to zero exponentially fast at infinity.
- We have no proof that $w_{\rm stat,\varepsilon}$ is positive.

Main results

Theorem 5 (E.Dumas–D.P., 2014)

For every $\tau_0 > 0$*, there exist positive constants* ε_0*,* δ_0 *and* C_0 *s.t. for all* $\varepsilon \in (0, \varepsilon_0)$*, when initial data* $w_{\mathrm{ini},\varepsilon} \in l^2(\mathbb{R})$ *satisfy*

$$\delta := \|w_{\mathrm{ini},\varepsilon} - w_{\mathrm{trav},\varepsilon}(0)\|_{l^2} \leq \delta_0,$$

then the unique solution w_ε *to the FPU lattice belongs to* $C^1([-\tau_0\varepsilon^{-3}, \tau_0\varepsilon^{-3}], l^2(\mathbb{Z}))$ *and satisfies*

$$\|w_\varepsilon(t) - w_{\mathrm{trav},\varepsilon}(t)\|_{l^2} \leq C_0\delta, \quad t \in \left[-\tau_0\varepsilon^{-3}, \tau_0\varepsilon^{-3}\right].$$

Main results

Theorem 5 (E.Dumas–D.P., 2014)

For every $\tau_0 > 0$, there exist positive constants ε_0, δ_0 and C_0 s.t. for all $\varepsilon \in (0, \varepsilon_0)$, when initial data $w_{\mathrm{ini},\varepsilon} \in l^2(\mathbb{R})$ satisfy

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Remarks:

- The travelling waves of the FPU lattice are stable w.r.t. modulations of any spatial scales, up to the time scale of $\mathcal{O}(\varepsilon^{-3})$.
- The constant C_0 may grow exponentially fast in τ_0.

Open questions:

- Convergence of the solitary wave in the FPU chain to the Gaussian wave in the log–KdV equation.
- Orbital stability of the Gaussian wave in the log–KdV equation.
- Development of numerical methods for the log–KdV equation.

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