Ground states of the energy super-critical Gross-Pitaevskii equation with harmonic potential

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Gross–Pitaevskii equation

The Gross-Pitaevskii theory in $\mathbb{R}^d$ with harmonic potential,

$$i\partial_t w = -\Delta w + |x|^2 w - |w|^{2p} w,$$

admits two conserved quantities of mass and energy,

$$M(w) = \int_{\mathbb{R}^d} |w|^2 dx, \quad E(w) = \int_{\mathbb{R}^d} \left(|\nabla w|^2 + |x|^2 |w|^2 - \frac{1}{p+1} |w|^{2p+2} \right) dx.$$

In the absence of harmonic potential, we adopt the following classification based on the scaling transformation:

$$w(t,x) \mapsto w_L(t,x) = L^{\frac{1}{p}} w(L^2 t, Lx), \quad L > 0,$$

which yields $M(w_L) = L^{\frac{2}{p} - d} M(w)$ and $E(w_L) = L^{\frac{2}{p} + 2 - d} E(w)$.

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- Mass-subcritical case ($dp < 2$): global existence in H^1
- Mass-critical case ($dp = 2$): global existence for small L^2 data and finite-time blow-up for large L^2
- Mass-supercritical case ($dp > 2$): global existence and scattering for $E(w) > 0$ and finite-time blow-up for $E(w) < 0$.

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- Energy-subcritical case: $(d-2)p < 2$.
- Energy-critical case: $(d-2)p = 2$, $d \geq 3$.
- Energy-supercritical case: $(d-2)p > 2$, $d \geq 3$.

We only consider the case $p = 1$ to simplify technical details so that $d = 4$ is the energy-critical case.

Standing wave solutions (bound states)

Standing wave solutions $w(t,x) = e^{-i\lambda t}u(x)$ satisfy the stationary Gross-Pitaevskii equation with harmonic potential:

$$-\Delta u + |x|^2 u - |u|^2 u = \lambda u,$$

Variationally, $u \in \mathcal{E} := H^1(\mathbb{R}^d) \cap L^{2,1}(\mathbb{R}^d) \cap L^4(\mathbb{R}^d)$ is a critical point of energy $E(u)$ subject to fixed mass $M(u)$, λ is Lagrange multiplier.

Among all bound states, we are only interested in the *ground state* with $u(x)$ satisfying:

- real and positive on $\mathbb{R}^d$;
- radially symmetric in $|x|$;
- bounded and monotonically decreasing to zero.

Such solutions bifurcate from $\lambda = d$ to $\lambda \lesssim d$.
No ground state solutions exist for $\lambda > d$.

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Energy-subcritical case $d \leq 3$:

- Existence for every $\lambda < d$ follows from variational theory due to compactness of embedding of $H^1(\mathbb{R}^d) \cap L^{2,1}(\mathbb{R}^d)$ into $L^4(\mathbb{R}^d)$ (Kavian & Weissler, 1994) (Fukuizumi, 2002)
- Uniqueness follows from ODE theory (Hirose & Ohta, 2002) (Hirose & Ohta, 2007)

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Energy-critical case $d = 4$:

- No solution exists for $\lambda < 0$ due to Pohozaev's identity
- Existence and uniqueness for some $\lambda \in (0, d)$ has been shown (Selem, 2011)
- It is still open if the solution exists as $\lambda \to 0$

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Energy-supercritical case $d \geq 5$:

- No solution exists for $\lambda < 0$ due to Pohozaev's identity
- The solution exists in a subset of $\lambda \in (0, d)$ (Selem & Kikuchi, 2012)
- The solution branch is connected to an unbounded solution $u_\infty \in \mathcal{E}$, $u_\infty \notin L^\infty$ for some $\lambda_\infty \in (0, d)$ (Selem & Kikuchi & Wei, 2013)

Shooting methods as a tool

The ground state is defined as a solution of the boundary-value problem for fixed $\lambda \in \mathbb{R}$:

$$\begin{cases} \mathfrak{u}''(r) + \frac{d-1}{r}\mathfrak{u}'(r) - r^2\mathfrak{u}(r) + \lambda\mathfrak{u}(r) + \mathfrak{u}(r)^3 = 0, \quad r > 0, \\ \mathfrak{u}(r) > 0, \qquad \mathfrak{u}'(r) < 0, \\ \lim_{r\to 0} \mathfrak{u}(r) < \infty, \quad \lim_{r\to\infty} \mathfrak{u}(r) = 0. \end{cases}$$

Solutions $\mathfrak{u}$ may not exist or their number may depend on λ.

The shooting method (Joseph & Lundgren, 1973) allows to find solutions $\mathfrak{u}$ from the initial-value problem:

$$\begin{cases} f_b''(r) + \frac{d-1}{r} f_b'(r) - r^2 f_b(r) + \lambda f_b(r) + f_b(r)^3 = 0, \quad r > 0, \\ f_b(0) = b, \quad f_b'(0) = 0, \end{cases}$$

where $b > 0$ is fixed parameter. If $f_b(r) > 0$, $f_b'(r) < 0$, and $f_b(r) \to 0$ as $r \to \infty$, then $\mathfrak{u}(r) = f_b(r)$ for some λ.

First result: existence

Theorem (BFPS, 2021)

Fix $d \geq 4$. For every $b > 0$, there exists $\lambda \in (d-4, d)$, labeled as $\lambda(b)$, such that the unique classical solution $f_b \in C^2(0,\infty)$ to the initial-value problem with $\lambda = \lambda(b)$ is a solution $\mathfrak{u} \in \mathcal{E} \cap L^\infty$ to the boundary-value problem.

- Uniqueness of $\lambda(b)$ is an open problem.
- This result holds both for critical and supercritical cases.

First result: existence

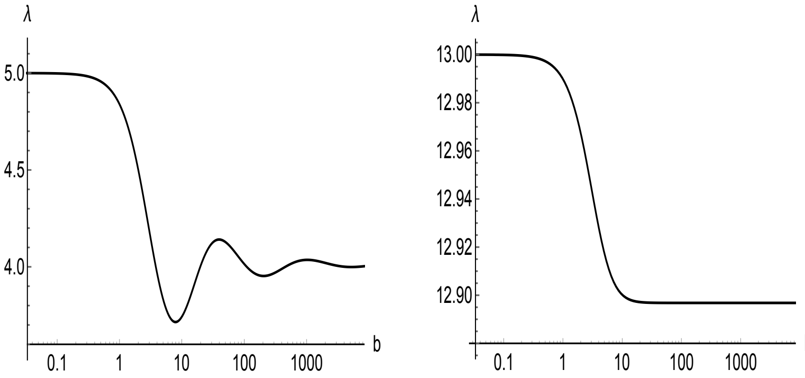

Figure 1: Graph of λ as a function of b for the ground state $\mathfrak{u}$ of the boundary-value problem for $d = 5$ (left) and $d = 13$ (right).

Ground state in the limit of $b \to \infty$?

The limiting singular solution $\mathfrak{u}_\infty \in \mathcal{E}$, $\mathfrak{u}_\infty \notin L^\infty$ is defined by

$$\mathfrak{u}_\infty(r) = \frac{\sqrt{d-3}}{r}\left[1 + \mathcal{O}(r^2)\right] \qquad \text{as} \quad r \to 0.$$

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Theorem (Selem–Kikuchi–Wei, 2013)

Fix $d \geq 5$. There exists $\lambda \in (0,d)$, labeled as λ_∞, such that the limiting singular solution $u_\infty \in \mathcal{E}$ exists so that $\lambda(b) \to \lambda_\infty$ and

$$\mathfrak{u}(b) \to \mathfrak{u}_\infty \quad \text{in} \quad \mathcal{E} \quad \text{as} \quad b \to \infty.$$

- Uniqueness of λ_∞ is an open problem.
- Details of convergence $\lambda(b) \to \lambda_\infty$ were not studied.

Second result: convergence

Theorem (BFPS, 2021)

Fix $d \geq 5$. Under some non-degeneracy assumptions, $\lambda(b)$ is uniquely defined near λ_∞ for $b \gg 1$ and

- $\lambda(b) - \lambda_\infty \sim A_\infty b^{-\beta} \sin(\alpha \ln b + \delta_\infty)$ *if* $5 \leq d \leq 12$, *for some* $A_\infty > 0$, $\delta_\infty \in (0, 2\pi)$, $\alpha > 0$, *and* $\beta > 0$
- $\lambda(b) - \lambda_\infty \sim B_\infty b^{-\kappa}$ *if* $d \geq 13$ *for some* $B_\infty \neq 0$ *and* $\kappa > 0$.

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The oscillatory behavior has been studied for the stationary NLS equation in a ball with dynamical system methods.

(Budd, Norbury, 1987), (Budd, 1989), (Merle & Peletier, 1991), (Dolbeault & Flores, 2007)

Linearization and Morse index

- Linearization around the ground state $\mathfrak{u}$:

$$\mathcal{L}_b := -\frac{d^2}{dr^2} - \frac{d-1}{r}\frac{d}{dr} + r^2 - \lambda(b) - 3\mathfrak{u}^2(r).$$

- Linearization around the singular solution $\mathfrak{u}_\infty$:

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$\mathcal{L}_b$ is well-defined in the form domain $\mathcal{E} := H^1_r \cap L^{2,1}_r$. It is a self-adjoint Sturm–Liouville operator in L^2_r with a purely point spectrum.

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Stability of standing waves in the Gross–Pitaevskii equation:

- $\mathfrak{u}$ is orbitally stable if $\mathcal{L}_b$ has exactly one negative eigenvalue and the mapping $\lambda \mapsto \|\mathfrak{u}\|_{L^2}^2$ is decreasing.
- $\mathfrak{u}$ is orbitally unstable if $\mathcal{L}_b$ has two or more negative eigenvalues

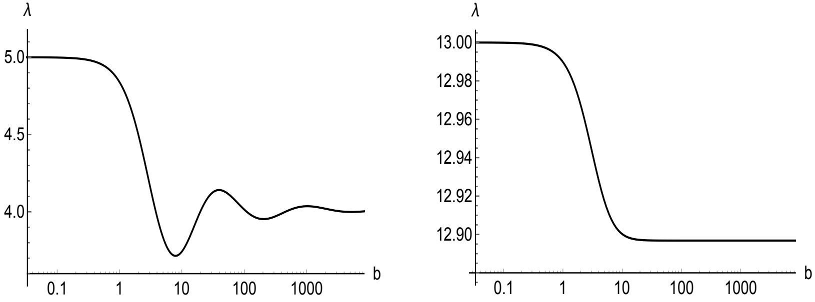
Note that $\langle \mathcal{L}_b \mathfrak{u}, \mathfrak{u} \rangle = -2\|\mathfrak{u}\|_{L_r^4}^4 < 0$, hence $\mathcal{L}_b$ is not positive.

Oscillatory versus monotone convergence

Since

$$\mathcal{L}_b \partial_b \mathfrak{u} = \lambda'(b)\mathfrak{u}, \quad \partial_b \mathfrak{u} \in \mathcal{E}_r,$$

the number of negative eigenvalues of $\mathcal{L}_b : \mathcal{E} \mapsto \mathcal{E}^*$ change for every b for which $\lambda'(b) = 0$.



Third result: stability

Theorem (P & Sobieszek, 2022)

For every $d \geq 13$, there exists $b_0 > 0$ such that the Morse index of $\mathcal{L}_b : \mathcal{E} \mapsto \mathcal{E}^*$ is finite and is independent of b for every $b \in (b_0, \infty)$. Moreover, it coincides with the Morse index of $\mathcal{L}_\infty : \mathcal{E} \mapsto \mathcal{E}^*$.

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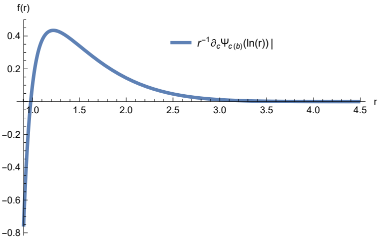
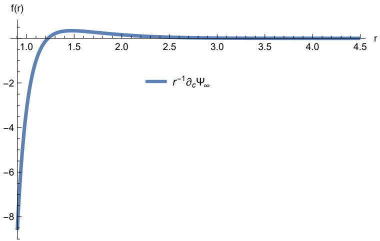
These approximations of $\mathcal{L}_b v = 0$ suggest that the Morse index is *one*.





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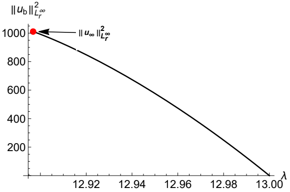
This graph suggests that the mapping $\lambda \mapsto \|\mathfrak{u}\|^2_{L^2}$ is *decreasing*.



Conclusion: the standing waves are stable for $d \geq 13$.

Emden-Fowler transformation

The initial-value problem,

$$\begin{cases} f_b''(r) + \frac{d-1}{r} f_b'(r) - r^2 f_b(r) + \lambda f_b(r) + f_b(r)^3 = 0, & r > 0, \\ f_b(0) = b, \quad f_b'(0) = 0, & \end{cases}$$

after the transformation

$$r = e^t, \qquad f(r) = \psi(t),$$

becomes the invariant manifold problem:

$$\begin{cases} \psi''(t) + (d-2)\psi'(t) + e^{2t}\left(\lambda + \psi(t)^2\right)\psi(t) - e^{4t}\psi(t) = 0, & t \in \mathbb{R}, \\ \psi(t) \to b, & t \to -\infty. \end{cases}$$

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The solution is a fixed point of the integral operator $A(\psi)$ given by

$$A(\psi)(t) := b + (d-2)^{-1} \int_{-\infty}^{t} [1 - e^{-(d-2)(t-t')}][e^{4t'}\psi - e^{2t'}(\lambda\psi + \psi^3)]dt'.$$

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There exists a unique solution $\psi \in C^2(\mathbb{R})$ such that

$$\psi_b(t) = b - (\lambda b + b^3)(2d)^{-1} e^{2t} + \mathcal{O}(e^{4t}), \quad \text{as } t \to -\infty.$$

Rigorously implemented shooting method

For the uniquely defined solution $\psi_b(t) = b + \mathcal{O}(e^{2t})$, we define the partition of $\mathbb{R} = I_+ \cup I_0 \cup I_-$ for parameter λ:

$$\begin{aligned}
I_+ &:= \{\lambda \in \mathbb{R} : \exists t_0 \in \mathbb{R} : \psi(t_0) = 0, \text{ while } \psi(t) > 0, \ \psi'(t) < 0, \ t < t_0\}, \\
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We have $I_- \cap I_+ = \emptyset$, $I_\pm \cap I_0 = \emptyset$, and furthermore,

- $[d, \infty) \subset I_+$ and I_+ is open;

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- $[d, \infty) \subset I_+$ and I_+ is open;
- $(-\infty, 0] \subset I_-$ and I_- is open;
- $I_0 \subset (0, d)$ is closed and if $\lambda(b) \in I_0$, then $\psi_b(t) \to 0$ as $t \to +\infty$ with the precise asymptotics:

$$\psi_b(t) \sim c e^{\frac{\lambda - d}{2} t} e^{-\frac{1}{2} e^{2t}}, \quad \text{as} \quad t \to +\infty,$$

  for some $c > 0$.

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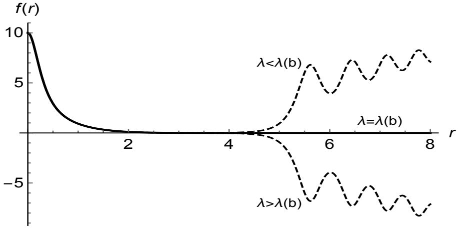

Towards the proof of convergence as $b \to \infty$

Recall the limiting singular solution $\mathfrak{u}_\infty \in \mathcal{E}$, $\mathfrak{u}_\infty \notin L^\infty$ defined by

$$\mathfrak{u}_\infty(r) = \frac{\sqrt{d-3}}{r}\left[1 + \mathcal{O}(r^2)\right] \qquad \text{as} \quad r \to 0.$$

The solution can be represented by $\mathfrak{u}(r) = r^{-1}F(r)$ with bounded F. Using Emden-Fowler transformation and $\psi(t) = e^{-t}\Psi(t)$, we obtain

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The limiting singular solution corresponds to the solution with

$$\Psi_\infty(t) = \sqrt{d-3} + \mathcal{O}(e^{2t}), \text{ as } t \to -\infty \quad \text{and} \quad \Psi_\infty(t) \to 0, \text{ as } t \to +\infty,$$

which exists for some $\lambda = \lambda_\infty$ (Selem–Kikuchi–Wei, 2013).

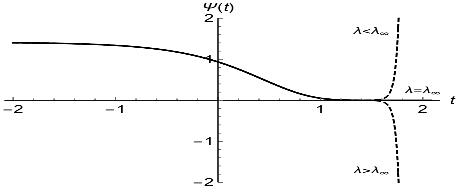
Towards the proof of convergence as $b \to \infty$

Recall the limiting singular solution $\mathfrak{u}_\infty \in \mathcal{E}$, $\mathfrak{u}_\infty \notin L^\infty$ defined by

$$\mathfrak{u}_\infty(r) = \frac{\sqrt{d-3}}{r}\left[1 + \mathcal{O}(r^2)\right] \qquad \text{as} \quad r \to 0.$$

The solution can be represented by $\mathfrak{u}(r) = r^{-1}F(r)$ with bounded F. Using Emden-Fowler transformation and $\psi(t) = e^{-t}\Psi(t)$, we obtain

$$\Psi''(t) + (d-4)\Psi'(t) + (3-d)\Psi(t) + \Psi(t)^3 + \lambda e^{2t}\Psi(t) - e^{4t}\Psi(t) = 0.$$



Two analytic family of solutions

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- The c-family $\Psi_c(t) \to 0$ as $t \to +\infty$ with

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- Their intersection for some $\lambda = \lambda(b)$ and $c = c(b)$:

$$\Psi_b(t) = \Psi_{c(b)}(t).$$

We want to prove: $\lambda(b) \to \lambda_\infty$ with some $c(b) \to c_\infty$ as $b \to +\infty$.

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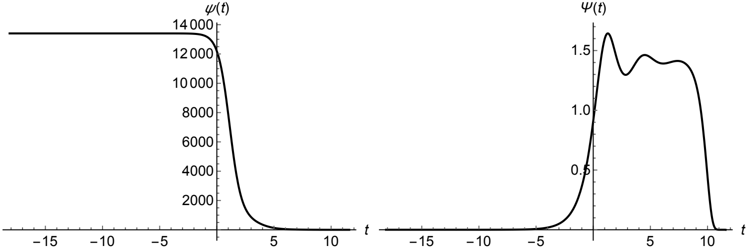
$$d = 5, \quad b = 14000:$$



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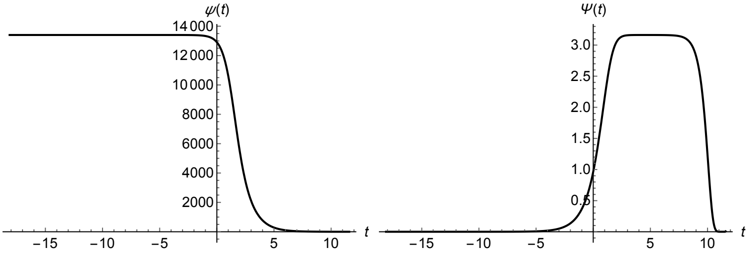
$$d = 13, \quad b = 14000:$$



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Formal truncation gives

$$\Theta''(t) + (d-4)\Theta'(t) + (3-d)\Theta(t) + \Theta(t)^3 = 0$$

with uniquely defined $\Theta(t) = e^t + \mathcal{O}(e^{3t})$ as $t \to -\infty$.

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Easy result for all large b:

$$\sup_{t\in(-\infty,0]} |\Psi_b(t - \log b) - \Theta(t)| \leq C_0 b^{-2}.$$

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Harder result for every $T > 0$ and $a \in (0,1)$:

$$\sup_{t\in[0,T+a\log b]} |\Psi_b(t-\log b) - \Theta(t)| \leq C_{T,a} b^{-2(1-a)}$$

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with uniquely defined $\Theta(t) = e^t + \mathcal{O}(e^{3t})$ as $t \to -\infty$.

$\Theta(t) \to \sqrt{d-3}$ as $t \to +\infty$ since

- $(\sqrt{d-3}, 0)$ is a stable spiral point for $5 \leq d \leq 12$
- $(\sqrt{d-3}, 0)$ is a stable nodal point for $d \geq 13$.

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with uniquely defined $\Theta(t) = e^t + \mathcal{O}(e^{3t})$ as $t \to -\infty$.

Non-degeneracy assumption ($5 \le d \le 12$):

$$\Theta(t) = \sqrt{d-3} + A_0 e^{-\beta t}\sin(\alpha t + \delta_0) + \mathcal{O}(e^{-2\beta t}) \quad \text{as} \quad t \to +\infty,$$

where $A_0 \neq 0$.

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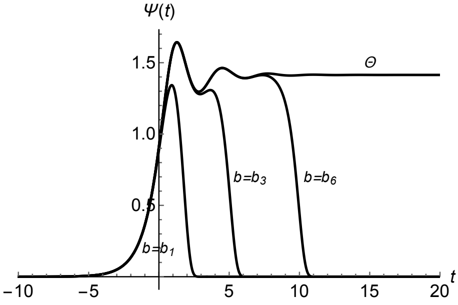

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Recall the limiting solution $\Psi_\infty(t) \to \sqrt{d-3}$ as $t \to -\infty$, which exists for $(\lambda, c) = (\lambda_\infty, c_\infty)$ and write

$$\Psi_c = \Psi_\infty + (\lambda - \lambda_\infty)\Psi_1 + (c - c_\infty)\Psi_2 + \Sigma,$$

for (λ, c) near $(\lambda_\infty, c_\infty)$.

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for (λ, c) near $(\lambda_\infty, c_\infty)$.

Easy result for every $t \in (-\infty, (a-1)\log b + T]$:

$$|\Psi_{1,2}(t) - A_{1,2} e^{-\beta t} \sin(\alpha t + \delta_{1,2})| \leq C_{T,a} b^{-2(1-a)} e^{-\beta t},$$

where $A_1, A_2 \neq 0$ (non-degeneracy assumption).

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Harder result for the remainder term for every $t \in [(a-1)\log b, 0]$:

$$|\Sigma(t)| \leq C_{T,a}\epsilon^2,$$

as long as $(\lambda - \lambda_\infty)^2 + (c - c_\infty)^2 \leq \epsilon^2 b^{-2\beta(1-a)}$ with small $\epsilon > 0$.

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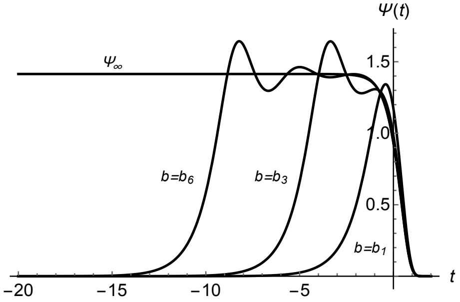

Intersection of the b-family and the c-family

We define $\lambda = \lambda(b)$ and $c = c(b)$ from

$$\Psi_b(t) = \Psi_{c(b)}(t), \quad t \in \mathbb{R}.$$

We can use the two asymptotic representations for every $t \in [(a-1)\log b, (a-1)\log b + T]$ with arbitrary $T > 0$.

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$$\begin{aligned} \Psi_b(T + (a-1)\log b) &= \Theta(T + a\log b) + \text{error} \\ &= \sqrt{d-3} + A_0 b^{-a\beta} e^{-\beta T} \sin(\alpha T + \delta_0) + \text{error} \end{aligned}$$

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$$\begin{aligned} \Psi_c(T + (a-1)\log b) &= \Psi_\infty(T + (a-1)\log b) + \text{linear terms} \\ &= \sqrt{d-3} + A_1(\lambda - \lambda_\infty) b^{(1-a)\beta} e^{-\beta T} \sin(\alpha T + \delta_1) \\ &\quad + A_2(c - c_\infty) b^{(1-a)\beta} e^{-\beta T} \sin(\alpha T + \delta_1) + \text{error} \end{aligned}$$

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Under the non-degeneracy assumption that $A_0, A_1, A_2 \neq 0$ we obtain with the implicit function theorem,

$$\lambda(b) - \lambda_\infty = A_\infty b^{-\beta} \sin(\alpha \log b + \delta_\infty) + \text{error},$$

inside $|\lambda - \lambda_\infty| \leq \epsilon b^{-\beta(1-a)}$.

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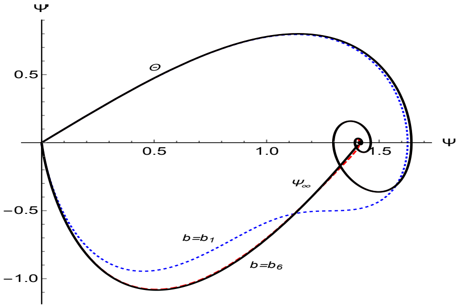

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Remarks

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- Derivative $\partial_c \Psi_c(t)$ is a solution of the linearized equation satisfying $\partial_c \Psi_c(t) \to 0$ as $t \to +\infty$.
- In the monotone case $d \geq 13$, under the non-degeneracy assumptions, we can show that if for $\lambda = \lambda(b)$,

$$\Psi_b(t) = \Psi_{c(b)}(t), \quad t \in \mathbb{R},$$

then there exists no $C \in \mathbb{R}$ such that

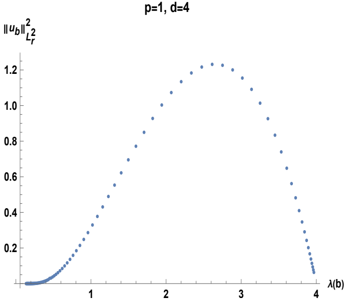
$$\partial_b \Psi_b(t) = C \partial_c \Psi_{c(b)}(t), \quad t \in \mathbb{R}.$$

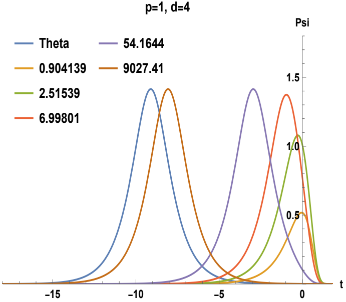
Hence the linearized operator $\mathcal{L}_b$ at $\mathfrak{u}_b$ has no zero eigenvalues.

Future goals

- We have shown existence of $\lambda(b)$ and λ_∞ but not uniqueness.
- No proof that if $\mathcal{L}_b$ has a zero eigenvalue in L^2_b, then $\lambda'(b) = 0$.
- In the oscillatory case, the Morse index is expected to increase by one every time $\lambda(b)$ passes through the extremal point.
- The existence of $\lambda(b)$ has been shown in the energy critical case $d = 4$ but we should prove that $\lambda(b) \to 0$ as $b \to \infty$ with the limiting singular solution being the algebraic soliton.

Energy-critical case $d = 4$





The b-family and c-family of solutions for $d = 4$

Consider the differential equation

$$\Psi''(t) - \Psi(t) + \Psi(t)^3 + \lambda e^{2t}\Psi(t) - e^{4t}\Psi(t) = 0,$$

for the solution $\Psi_b(t) = be^t + \mathcal{O}(e^{3t})$ as $t \to -\infty$.

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Formal truncation gives

$$\Theta''(t) - \Theta(t) + \Theta(t)^3 = 0$$

with uniquely defined

$$\Theta(t) = \frac{8be^t}{8 + b^2 e^{2t}},$$

which corresponds to the algebraic soliton in variable $r = e^t$.

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One can again continue the solution for every $T > 0$ and $a \in (0, 1)$:

$$\sup_{t\in[0,T+a\log b]} |\Psi_b(t - \log b) - \Theta(t)| \leq C_{T,a} b^{-2(1-a)},$$

such that $\Theta(T + a\log b) = 8b^{-a}e^{-T} + \mathcal{O}(b^{-3a})$ is small.

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For the c-family, we can take the solution of the linear equation

$$\Psi_c(t) = ce^t e^{-\frac{1}{2}e^{2t}} U(e^{2t}; a = 1 - \frac{\lambda}{4}, b = 2),$$

where $U(z; a, b)$ is the Tricomi solution of Kummer's differential equation.

The b-family and c-family of solutions for $d = 4$

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for the solution $\Psi_b(t) = be^t + \mathcal{O}(e^{3t})$ as $t \to -\infty$.

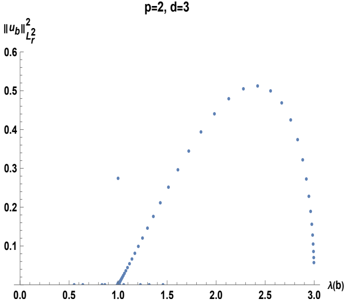
Both solutions Ψ_b and Ψ_c are defined for arbitrary λ and their intersection is tangential in the sense that equation

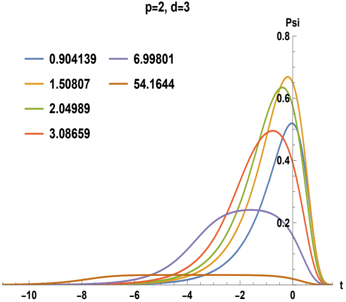
$$\Psi_b(t) = \Psi_{c(b)}(t), \quad t \in \mathbb{R},$$

determines only $c(b) = 8b^{-1} + \mathrm{o}(b^{-1})$ but not $\lambda(b)$.

Energy-critical case $d = 3$ (quintic nonlinearity)

The situation becomes even more interesting in three dimensions:
$f_b'' + 2r^{-2}f_b' - r^2 f_b + \lambda f_b + f_b^5 = 0$





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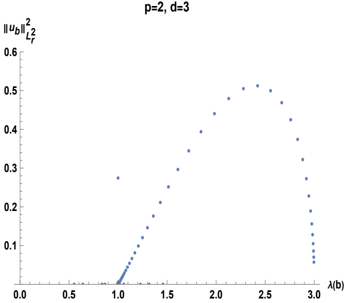

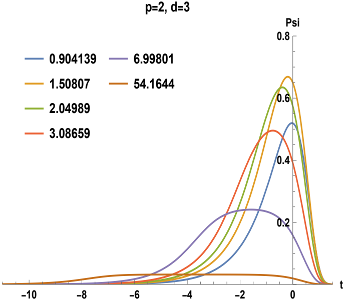

Thank you for attention!