Transverse instabilities of deep-water solitons

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Formulation of the problem

Hyperbolic two-dimensional NLS equation

$$i\psi_t + \psi_{xx} - \psi_{yy} + 2|\psi|^2\psi = 0,$$

where $\psi : \mathbb{R}^2 \times \mathbb{R}_+ \mapsto \mathbb{C}$ is the envelope amplitude and $\eta = \mathrm{Re}(\psi e^{ik_0 x - i\omega_0 t})$ is the elevation of the water wave surface.

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A line soliton solution

$$\psi = e^{it}\mathrm{sech}(x), \qquad \eta = \cos(k_0 x - (\omega_0 - 1)t)\mathrm{sech}(x),$$

where four parameters (phase translation, coordinate translation, frequency, velocity) can be constructed by using the Lie point symmetries of the NLS equation.

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Question: Is the line soliton stable with respect to transverse (y-dependent) perturbations?

Linearized stability and instability

Consider a linear perturbation to the line soliton

$$\psi = \left(\mathrm{sech}(x) + \epsilon u(x,y,t) + i\epsilon v(x,y,t)\right) e^{it}$$

and find the linear PDEs:

$$-u_t = v_{xx} - v_{yy} + (2\mathrm{sech}^2 x - 1)v, \;\; v_t = u_{xx} - u_{yy} + (6\mathrm{sech}^2 x - 1)u$$

Use the Fourier transform in y and Laplace transform in t, e.g.

$$u = U(x)e^{i\rho y}e^{\Omega t}, \;\; v = V(x)e^{i\rho y}e^{\Omega t}, \qquad \rho \in \mathbb{R}, \;\; \Omega \in \mathbb{C}$$

where $(U,W) \in L^2(\mathbb{R})$ is an eigenvector of Schrödinger operators

$$\Omega U = (L_- - \rho^2)V, \qquad -\Omega V = (L_+ - \rho^2)U.$$

The line soliton is transversely unstable if $\mathrm{Re}(\Omega) > 0$ for $\rho \neq 0$.

History of analytical studies

- Zakharov–Rubenchik (1974), Yajima (1974): splitting of zero eigenvalues for small ρ into a pair of real eigenvalues $\pm\Omega_1$ and a pair of purely imaginary eigenvalues $\pm\Omega_2$:

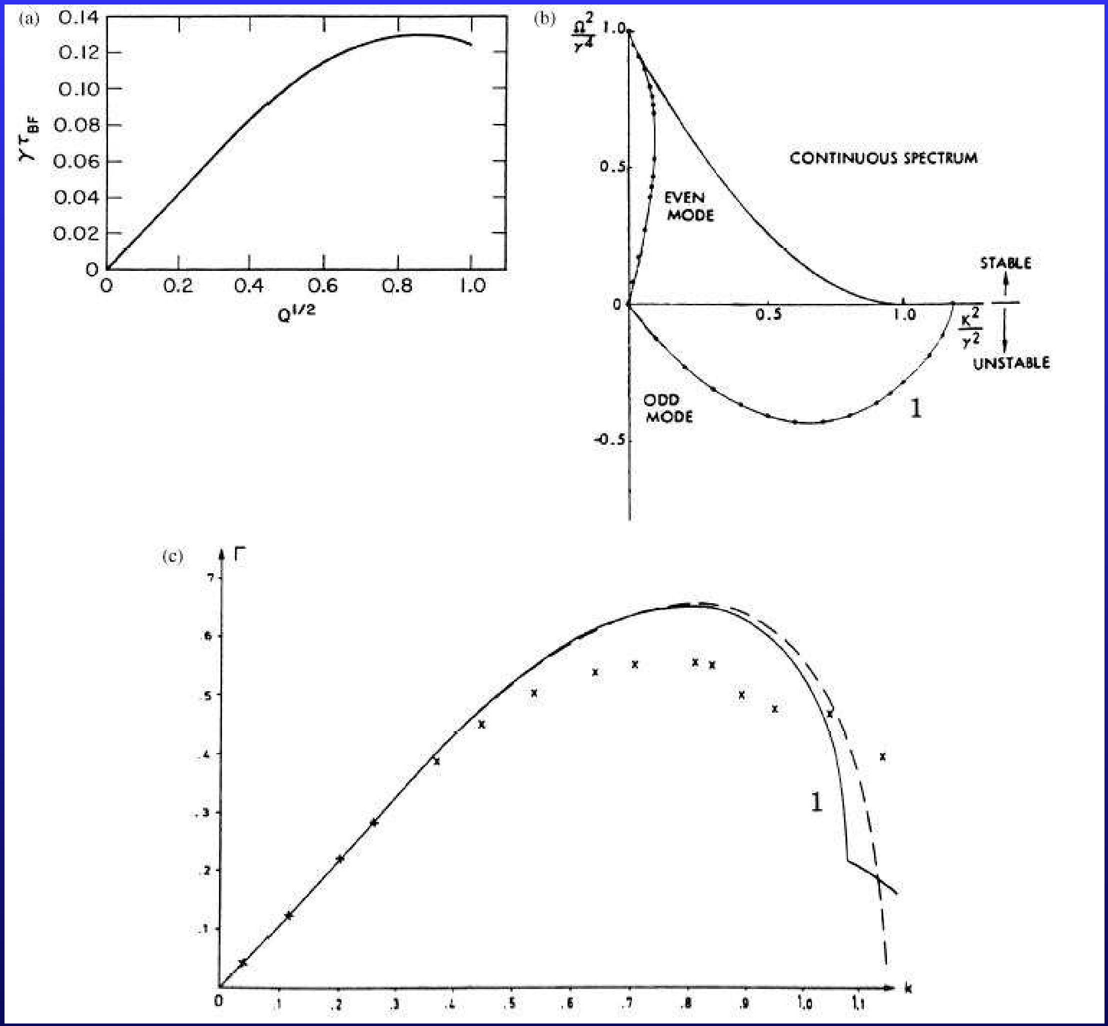
$$\Omega_1^2 = \frac{4}{3}\rho^2 + \mathrm{O}(\rho^4), \qquad \Omega_2^2 = -4\rho^2 + \mathrm{O}(\rho^4).$$

- Ablowitz–Segur (1979): no real eigenvalues exist for large ρ as $\Omega^2 = -\rho^4 + \mathrm{O}(\rho^2)$.

- Anderson et al. (1979): existence of complex eigenvalues for values of $\rho \sim 1$ in the Rayleigh–Ritz variational method.

- Kivshar–Pelinovsky (2000): nonlinear theory of break-up of line soliton into dispersive clusters.

History of numerical studies



Top left: Cohen et al. (1976). Top right: Saffman–Yuen (1978). Bottom: Anderson et al. (1979).

Summary of our new results

- Numerical analysis of **all** isolated eigenvalues and their bifurcations
- Prediction of instabilities for **any** $\rho > 0$.
- Analytical **proof** of existence of unstable eigenvalues for $0 < \rho < 1$
- Rigorous **analysis** of bifurcations of unstable eigenvalues from $\rho = 1$ to $\rho > 1$.

B. Deconinck, D.P, J. Carter, Proc. Roy. Soc. A **462**, 2039 (2006)

D.P., Math. Comp. Simul. **55**, 585 (2001)

Numerical Hill's method

Consider the spectral problem

$$\phi_x = (A(x) + \lambda)\,\phi, \qquad \phi \in \mathbb{C}^n$$

where $A(x + L) = A(x)$ and $\lambda \in \mathbb{C}$. We are looking for eigenvalues λ when eigenvector ϕ is in $L^\infty([0, L])$ space.

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Floquet's Theorem: There exists a constant $n \times n$ matrix R and L-periodic $n \times n$ matrix P, such that the fundamental matrix solution $\phi(x)$ is $\phi(x) = P(x)e^{Rx}$.

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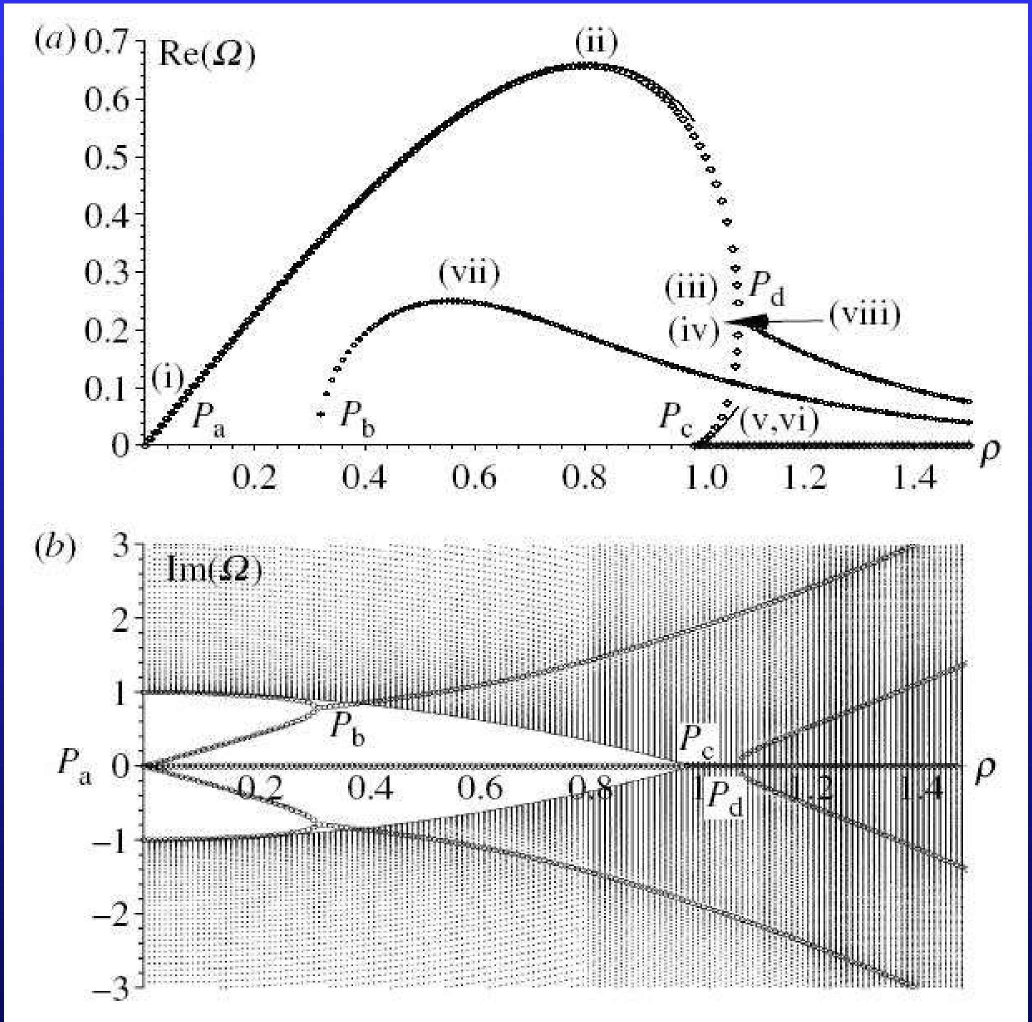
The bounded eigenvector can be decomposed as

$$\phi(x) = e^{i\mu x} \sum_{k \in \mathbb{Z}} \phi_k e^{2\pi i x/L},$$

where $\mu \in \left[-\frac{\pi}{L}, \frac{\pi}{L}\right]$. For each value of μ, the spectrum of λ can be found by truncation of Fourier series.

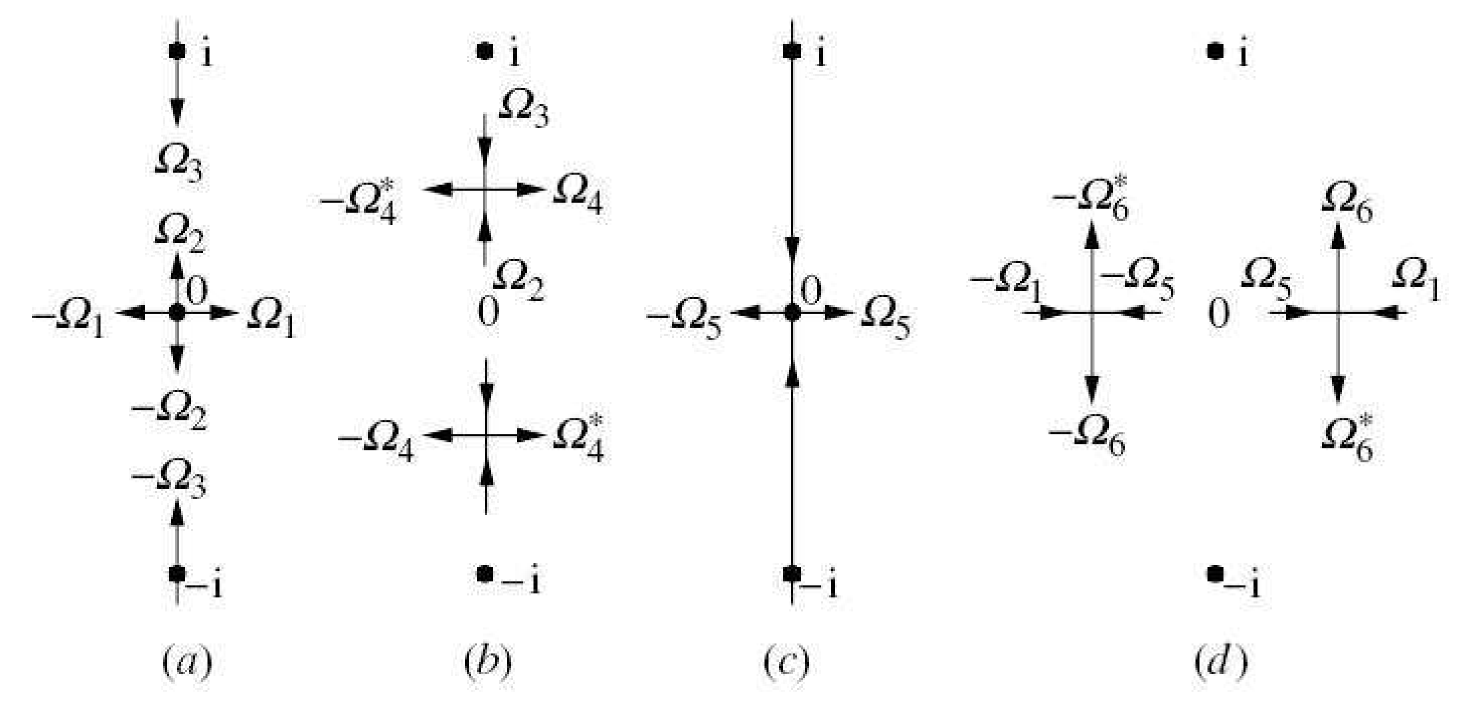
Numerical analysis of all eigenvalues



Conclusion: Line soliton is unstable for any $\rho > 0$

Bifurcations: $\rho = 0$, $\rho \approx 0.31$, $\rho = 1$, $\rho \approx 1.08$.

Numerical analysis of all bifurcations



Bifurcations:

(a) $\rho = 0$ - two pairs arise from multiple eigenvalue $\Omega = 0$
one pair arises from the end point of the continuous spectrum

(b) $\rho \approx 0.31$ - Hamiltonian-Hopf bifurcation

(c) $\rho = 1$ - collision of end points of the continuous spectrum

(d) $\rho \approx 1.08$ - double real eigenvalue bifurcation

Proof of existence of unstable eigenvalues

Consider the spectral problem

$$(L_+ - \rho^2)U = -\Omega V, \qquad (L_- - \rho^2)V = \Omega U$$

$$L_+ = -\partial_x^2 + 1 - 6\text{sech}^2 x, \qquad L_- = -\partial_x^2 + 1 - 2\text{sech}^2 x.$$

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Spectra $\sigma(L_\pm)$ in $L^2(\mathbb{R})$:

- $\sigma(L_+)$ - two isolated eigenvalues at $\sigma = -3$ and $\sigma = 0$ and continuous spectrum for $\sigma \geq 1$
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For $0 < \rho < 1$, the spectral problem is equivalent to the generalized eigenvalue problem

$$\left(L_+ - \rho^2\right) U = \gamma \left(L_- - \rho^2\right)^{-1} U, \qquad \gamma = -\Omega^2$$

Sylvester–Pontryagin–Grillakis Theorem

Theorem: Let L and M be a self-adjoint operators in $L^2(\mathbb{R})$ with finitely many negative eigenvalues $n(L)$ and $n(M)$ and empty kernels. Then, there are exactly $n(L)$ and $n(M)$ eigenvalues γ of $Lu = \gamma M u$ in $L^2(\mathbb{R})$ such that $(u, Lu) \leq 0$ and $(u, Mu) \leq 0$.

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Application:

$$\left(L_+ - \rho^2\right) U = \gamma \left(L_- - \rho^2\right)^{-1} U, \qquad \gamma = -\Omega^2$$

For $0 < \rho < 1$, there exist two eigenvalues γ such that $(U, (L_+ - \rho^2)U) \leq 0$ and one eigenvalue γ such that $(U, (L_- - \rho^2)^{-1} U) \leq 0$.

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Conclusion: One eigenvalue $\Omega \in \mathbb{R}_+$ is always unstable for $0 < \rho < 1$, and two eigenvalues are either purely imaginary for $0 < \rho < 0.31$ or complex for $0.31 < \rho < 1$.

Bifurcations of unstable eigenvalues

Let $\rho^2 = 1 - \frac{\kappa_+^2 + \kappa_-^2}{2}$, $\Omega = \frac{\kappa_+^2 - \kappa_-^2}{2\mathrm{i}}$ and rewrite the eigenvalue problem

$$\begin{aligned} \left(-\partial_x^2 - 6\mathrm{sech}^2 x\right) U &= -\frac{1}{2}(\kappa_+^2 + \kappa_-^2)U + \frac{i}{2}(\kappa_+^2 - \kappa_-^2)V, \\ \left(-\partial_x^2 - 2\mathrm{sech}^2 x\right) V &= -\frac{i}{2}(\kappa_+^2 - \kappa_-^2)U - \frac{1}{2}(\kappa_+^2 + \kappa_-^2)V, \end{aligned}$$

Bifurcations of unstable eigenvalues

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- Bifurcation point $\rho = 1$, $\Omega = 0$ corresponds to $(\kappa_+, \kappa_-) = (0, 0)$
- Two solutions $\mathbf{u}_\pm(x)$ decay like $e^{\kappa_\pm x}$ as $x \to -\infty$ and two solutions $\mathbf{v}_\pm(x)$ decays like $e^{-\kappa_\pm x}$ as $x \to +\infty$ in the domain $\mathrm{Re}(\kappa_\pm) > 0$
- The coordinates $(\kappa_+, \kappa_-) \in \mathbb{C}^2$ unfold the branch point singularity in coordinates $\Omega \in \mathbb{C}$ and $\rho^2 - 1 \in \mathbb{R}$.

Evans function

Let $\mathbf{u}_\pm(x)$ and $\mathbf{v}_\pm(x)$ be fundamental solutions in the domain $\mathrm{Re}(\kappa_\pm) > 0$ and define the Wronskian determinant:

$$E(\kappa_+, \kappa_-) = \det\left(\mathbf{u}_+(x), \mathbf{u}_-(x), \mathbf{v}_+(x), \mathbf{v}_-(x)\right)\big|_{x=0}.$$

Zeros of $E(\kappa_+, \kappa_-)$ coincide with eigenvalues of the original problem in the domain $\mathrm{Re}(\kappa_\pm) > 0$.

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- Fundamental solutions $\mathbf{u}_\pm(x)$, $\mathbf{v}_\pm(x)$, and their determinant $E(\kappa_+, \kappa_-)$ are analytic functions near $(\kappa_+, \kappa_-) = (0, 0)$.
- The Taylor series expansion holds near $(0, 0)$:

$$E(\kappa_+, \kappa_-) = -4(\kappa_+ + \kappa_-)^2 + 10(\kappa_+ + \kappa_-)^3 - 13(\kappa_+^4 + \kappa_-^4)$$
$$-51(\kappa_+^2 + \kappa_-^2)\kappa_+\kappa_- - 72\kappa_+^2\kappa_-^2 - \alpha_0(\kappa_+^2 - \kappa_-^2)^2 + \mathrm{O}(5),$$

where α_0 is a numerical coefficient.

Asymptotic result

Let $\alpha = \kappa_+ + \kappa_-$ and $\beta = \kappa_+ - \kappa_-$, such that

$$E(\alpha, \beta) = -4\alpha^2 + 10\alpha^3 - \frac{25}{2}\alpha^4 + \frac{1}{4}\beta^4 - \left(\alpha_0 + \frac{3}{4}\right)\alpha^2\beta^2 + \mathrm{O}(5).$$

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By the Newton polygon technique, there exists only one family of zeros of $E(\alpha, \beta) = 0$ in a neighborhood of $(\alpha, \beta) = (0, 0)$ such that

$$\alpha^2 = \frac{1}{16}\beta^4 + \mathrm{O}(\beta^6).$$

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Restoring the original variables

$$\kappa_\pm = \sqrt{1 - \rho^2 \pm i\Omega},$$

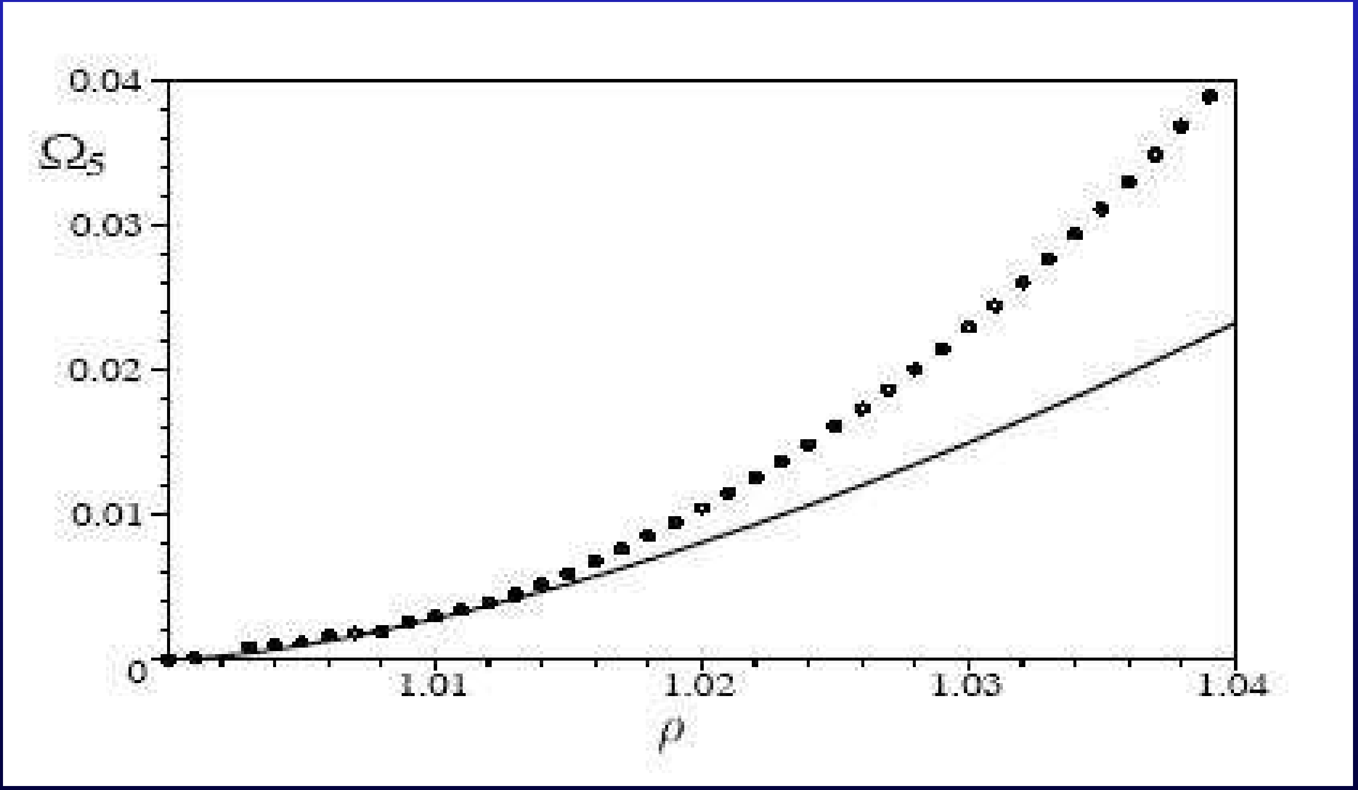
and performing a careful analysis of branches of the square root function, we obtain the final asymptotic result.

Bifurcation of eigenvalue Ω_5

Asymptotic result

$$\Omega_5 = 2\sqrt{2}\,(\rho - 1)^{3/2} + \mathrm{O}((\rho - 1)^{5/2})$$

Numerical result



Conclusions

- Earlier analytical results on $\rho \ll 1$, $\rho \gg 1$, and $\rho \sim 1$ are confirmed numerically by using the Hill's method
- Instabilities are rigorously proved for $0 < \rho < 1$ with the count of eigenvalues in Pontryagin spaces
- Bifurcations of collision of end points of the continuous spectrum is rigorously analyzed with the use of the Evans function and a new unfolding technique.