Periodic and double periodic waves in NLS: existence and stability

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The focusing NLS equation

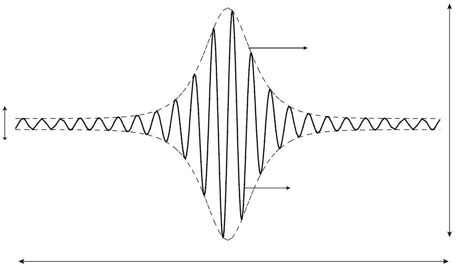
The focusing nonlinear Schrödinger (NLS) equation

$$i\psi_t + \frac{1}{2}\psi_{xx} + |\psi|^2\psi = 0$$

has been derived as the main model for modulating quasi-harmonic waves

$$\epsilon\psi(\epsilon(x-ct),\epsilon^2 t)e^{i(k_0 x-\omega_0 t)} + \epsilon\bar{\psi}(\epsilon(x-ct),\epsilon^2 t)e^{-i(k_0 x-\omega_0 t)} + \text{higher-order terms}$$

from water wave equations, Maxwell equations, and the like.



$\psi(x,t) = e^{it}$ is the constant-amplitude wave,
$\psi(x,t) = \mathrm{sech}(x)e^{it/2}$ is a solitary wave.

The rogue wave of the cubic NLS equation

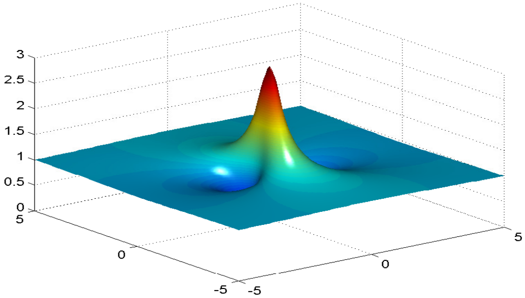
The focusing nonlinear Schrödinger (NLS) equation

$$i\psi_t + \frac{1}{2}\psi_{xx} + |\psi|^2\psi = 0$$

admits the exact solution

$$\psi(x,t) = \left[1 - \frac{4(1+2it)}{1+4x^2+4t^2}\right] e^{it}.$$

It was discovered by H. Peregrine (1983) and was labeled as *the rogue wave*.



Modulational instability of the constant-amplitude wave

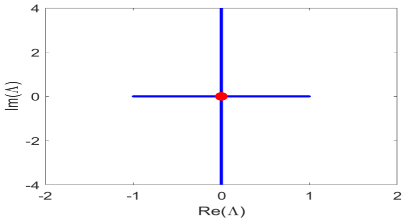
The rogue wave solution is related to the modulational instability of the constant-amplitude wave:

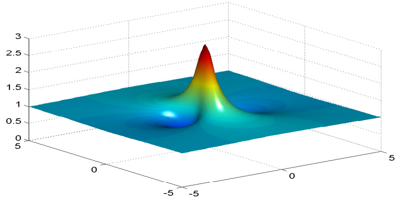
$$\psi(x,t) = e^{it}\left[1 + (k^2 + 2i\Lambda)e^{\Lambda t + ikx} + (k^2 + 2i\bar{\Lambda})e^{\bar{\Lambda} t - ikx}\right],$$

where $k \in \mathbb{R}$ is the wave number and Λ is given by

$$\Lambda^2 = k^2\left(1 - \frac{1}{4}k^2\right).$$

The wave is unstable for $k \in (0,2)$.



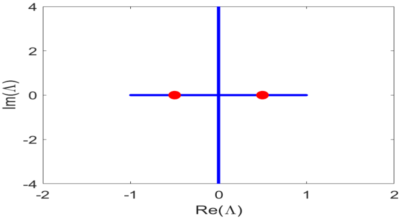


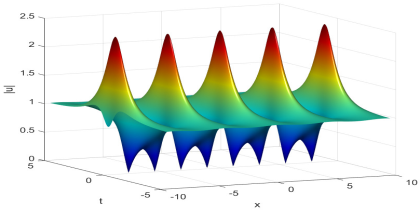
Other rogue waves - Akhmediev breathers (AB)

Spatially periodic homoclinic solution was constructed by N.N. Akhmediev, V.M. Eleonsky, and N.E. Kulagin (1985):

$$\psi(x,t) = e^{it}\left[1 - \frac{2(1-\lambda^2)\cosh(k\lambda t) + ik\lambda\sinh(k\lambda t)}{\cosh(k\lambda t) - \lambda\cos(kx)}\right],$$

where $k = 2\sqrt{1-\lambda^2} \in (0,2)$ and $\lambda \in (0,1)$ is the only free parameter. There is a unique solution for each spatial period $L = \frac{2\pi}{k} = \frac{\pi}{\sqrt{1-\lambda^2}} > \pi$.



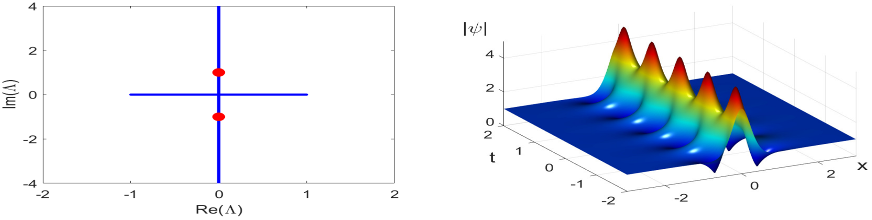


Other rogue waves - Kuznetsov-Ma breathers

Temporally periodic soliton was constructed by E. A. Kuznetsov (1977) and Y.-C. Ma (1979):

$$\psi(x,t) = \left[1 - \frac{2(\lambda^2-1)\cos(\beta\lambda t) + i\beta\lambda\sin(\beta\lambda t)}{\lambda\cosh(\beta x) - \cos(\beta\lambda t)}\right] e^{it},$$

where $\beta = 2\sqrt{\lambda^2-1}$ and $\lambda \in (1,\infty)$ is the only free parameter. There is a unique solution for each temporal period $T = \frac{2\pi}{\beta\lambda} = \frac{\pi}{\lambda\sqrt{\lambda^2-1}} > 0$ with $k = i\beta$.



Traveling periodic waves

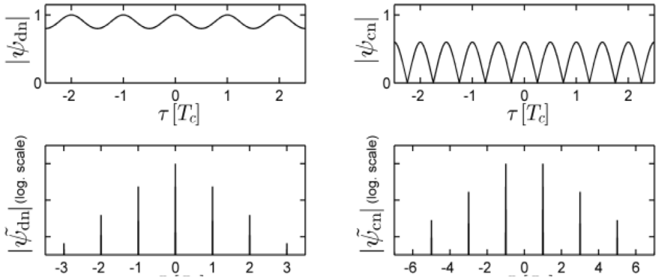
The focusing nonlinear Schrödinger (NLS) equation

$$i\psi_t + \frac{1}{2}\psi_{xx} + |\psi|^2\psi = 0$$

also admits the periodic traveling and standing wave solutions, e.g. the dnoidal and cnoidal waves:

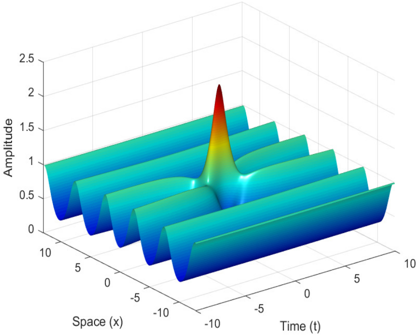
$$\psi_{\rm dn}(x,t) = {\rm dn}(x;k)e^{i(1-k^2/2)t}, \quad \psi_{\rm cn}(x,t) = k{\rm cn}(x;k)e^{i(k^2-1/2)t},$$

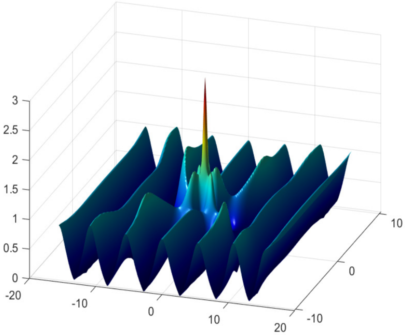
where $k \in (0,1)$ is elliptic modulus.



Rogue wave on background of periodic waves

J. Chen, D. P., Proceedings A (2018)
J. Chen, D. P., R. White, Physica D (2020)





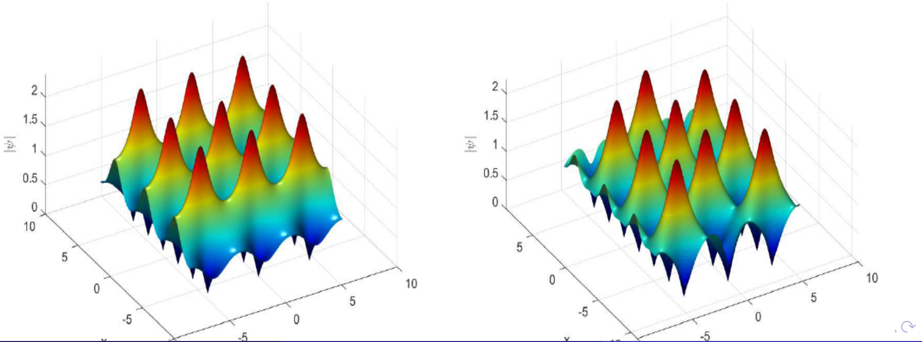
Double-periodic wave background

Double-periodic solutions (Akhmediev, Eleonskii, Kulagin, 1987):

$$\psi(x,t) = k\frac{\mathrm{cn}(t;k)\mathrm{cn}(\sqrt{1+k}x;\kappa) + i\sqrt{1+k}\mathrm{sn}(t;k)\mathrm{dn}(\sqrt{1+k}x;\kappa)}{\sqrt{1+k}\mathrm{dn}(\sqrt{1+k}x;\kappa) - \mathrm{dn}(t;k)\mathrm{cn}(\sqrt{1+k}x;\kappa)}e^{it},$$

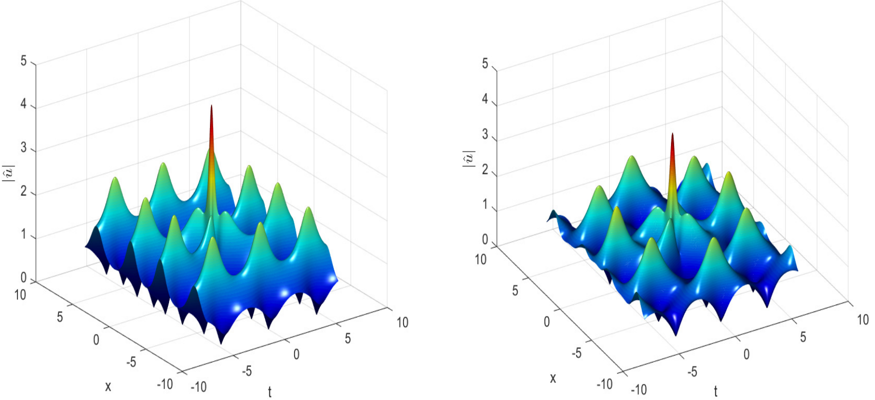
$$\psi(x,t) = \frac{\mathrm{dn}(t;k)\mathrm{cn}(\sqrt{2}x;\kappa) + i\sqrt{k(1+k)}\mathrm{sn}(t;k)}{\sqrt{1+k} - \sqrt{k}\mathrm{cn}(t;k)\mathrm{cn}(\sqrt{2}x;\kappa)}e^{ikt},$$

where $k \in (0,1)$ is elliptic modulus and $\kappa \in (0,1)$ is determined by k.



Rogue wave on background of double-periodic waves

J. Chen, D. P., R. White, Phys. Rev. E (2019)



NLS hierarchy

The focusing nonlinear Schrödinger (NLS) equation

$$i\psi_t + \frac{1}{2}\psi_{xx} + |\psi|^2\psi = 0$$

is a member of the NLS hierarchy

$$\frac{d}{dt_k}\left[\begin{array}{c} u \\ \bar{u} \end{array}\right] = J\nabla H_k(u), \quad \nabla H_{k+1}(u) = R\nabla H_k(u),$$

where

$$J = i\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad R = i\begin{bmatrix} \partial_x + 2\bar{u}\partial_x^{-1}u & -2\bar{u}\partial_x^{-1}\bar{u} \\ 2u\partial_x^{-1}u & -\partial_x - 2u\partial_x^{-1}\bar{u} \end{bmatrix},$$

Thus, we obtain

$$H_0 = \int_{\mathbb{R}} |u|^2 dx, \quad H_1 = \frac{i}{2}\int_{\mathbb{R}} (u\bar{u}_x - u_x\bar{u})\, dx,$$

$$H_2 = \int_{\mathbb{R}} \left(|u_x|^2 - |u|^4\right) dx, \quad H_3 = \frac{i}{2}\int_{\mathbb{R}} \left[u_x\bar{u}_{xx} - u_{xx}\bar{u}_x - 3|u|^2(u\bar{u}_x - u_x\bar{u})\right] dx.$$

Stationary Lax-Novikov equations

The stationary (Lax–Novikov) equations are given by

$$\begin{aligned}
&\nabla H_1(u) + 2c\nabla H_0(u) = 0,\\
&\nabla H_2(u) + 2c\nabla H_1(u) + 4b\nabla H_0(u) = 0,\\
&\nabla H_3(u) + 2c\nabla H_2(u) + 4b\nabla H_1(u) + 8a\nabla H_0(u) = 0,
\end{aligned}$$

or explicitly,

$$\begin{aligned}
&u'(x) + 2icu = 0,\\
&u''(x) + 2|u|^2 u + 2icu' + 4bu = 0,\\
&u'''(x) + 6|u|^2 u' + 2ic(u'' + 2|u|^2 u) + 4bu' + 8iau = 0,
\end{aligned}$$

where c, b, a are constants.

Solutions of stationary Lax-Novikov equations

In terms of the NLS equation

$$i\psi_t + \frac{1}{2}\psi_{xx} + |\psi|^2\psi = 0$$

the stationary Lax–Novikov equations

$$\begin{aligned} & u' + 2icu = 0, \\ & u'' + 2|u|^2 u + 2icu' + 4bu = 0, \\ & u''' + 6|u|^2 u' + 2ic(u'' + 2|u|^2 u) + 4bu' + 8iau = 0, \end{aligned}$$

generate the following solutions:

1. Constant-amplitude wave $\psi(x,t) = Ae^{-2ic(x+ct)+iA^2 t}$,
2. Traveling standing wave $\psi(x,t) = u(x+ct)e^{-2ibt}$
3. Double-periodic wave $\psi(x,t) = [q(x,t) + i\delta(t)]e^{it+i\alpha(t)}$, where $q(x+L,t) = q(x,t+T) = q(x,t)$, $\delta(t+T) = \delta(t)$, $\alpha(t+T) = \alpha(t)$.

Characterization of $u'' + 2|u|^2 u + 2icu' + 4bu = 0$

Consider the Lax system of linear equations

$$\varphi_x = U(\lambda, u)\varphi, \qquad U(\lambda, u) = \begin{pmatrix} \lambda & u \\ -\bar{u} & -\lambda \end{pmatrix}$$

and

$$\varphi_t = V(\lambda, u)\varphi, \qquad V(\lambda, u) = i \begin{pmatrix} \lambda^2 + \frac{1}{2}|u|^2 & \frac{1}{2}u_x + \lambda u \\ \frac{1}{2}\bar{u}_x - \lambda \bar{u} & -\lambda^2 - \frac{1}{2}|u|^2 \end{pmatrix}.$$

Fix $\lambda = \lambda_1 \in \mathbb{C}$ with $\varphi = (p_1, q_1) \in \mathbb{C}^2$ and set $u = p_1^2 + \bar{q}_1^2$. The spectral problem $\varphi_x = U(\lambda, u)\varphi$ becomes the Hamiltonian system generated by

$$H = \lambda_1 p_1 q_1 + \bar{\lambda}_1 \bar{p}_1 \bar{q}_1 + \frac{1}{2}(p_1^2 + \bar{q}_1^2)(\bar{p}_1^2 + q_1^2).$$

with additional constant $F = i(p_1 q_1 - \bar{p}_1 \bar{q}_1)$.

(Cao–Geng, 1990) (Cao–Wu–Geng, 1999) (R.Zhou, 2009) (Chen–P, 2018)

Second-order Lax–Novikov equation

By differentiating of the constraints between u and (p_1, q_1), we obtain

$$\begin{aligned} u &= p_1^2 + \bar{q}_1^2, \\ u' + 2iFu &= 2(\lambda_1 p_1^2 - \bar{\lambda}_1 \bar{q}_1^2), \\ u'' + 2|u|^2 u + 2iFu' - 4Hu &= 4(\lambda_1^2 p_1^2 + \bar{\lambda}_1^2 \bar{q}_1^2), \end{aligned}$$

which yields the second-order Lax–Novikov equation:

$$u'' + 2|u|^2 u + 2icu' + 4bu = 0,$$

where $c := F + i(\lambda_1 - \bar{\lambda}_1)$ and $b := -H - iF(\lambda_1 - \bar{\lambda}_1) - |\lambda_1|^2$.

The second-order equation admits two conserved quantities:

$$\begin{aligned} i(u'\bar{u} - u\bar{u}') - 2c|u|^2 &= 4a, \\ |u'|^2 + |u|^4 + 4b|u|^2 &= 8d. \end{aligned}$$

Integrability of the Hamiltonian system

The Hamiltonian system for (p_1, q_1) is obtained from the Lax equation

$$\frac{d}{dx}W(\lambda) = U(\lambda,u)W(\lambda) - W(\lambda)U(\lambda,u),$$

where $U(\lambda,u)$ is defined under the constraint $u = p_1^2 + \bar{q}_1^2$ and

$$W(\lambda) = \begin{pmatrix} W_{11}(\lambda) & W_{12}(\lambda) \\ \bar{W}_{12}(-\lambda) & -\bar{W}_{11}(-\lambda) \end{pmatrix},$$

with

$$W_{11}(\lambda) = 1 - \frac{p_1 q_1}{\lambda - \lambda_1} + \frac{\bar{p}_1 \bar{q}_1}{\lambda + \bar{\lambda}_1}, \quad W_{12}(\lambda) = \frac{p_1^2}{\lambda - \lambda_1} + \frac{\bar{q}_1^2}{\lambda + \bar{\lambda}_1}.$$

Due to relations between u and p_1^2, $\bar{q}_1^2$, and $p_1 q_1$, we have

$$W_{11}(\lambda) = \frac{\lambda^2 + ic\lambda + b + \frac{1}{2}|u|^2}{(\lambda - \lambda_1)(\lambda + \bar{\lambda}_1)}, \quad W_{12}(\lambda) = \frac{u\lambda + icu + \frac{1}{2}u'}{(\lambda - \lambda_1)(\lambda + \bar{\lambda}_1)}.$$

Algebraic polynomial for $u'' + 2|u|^2 u + 2icu' + 4bu = 0$

$\det W(\lambda)$ is constant in (x,t) and has simple poles at λ_1 and $-\bar{\lambda}_1$:

$$\det[W(\lambda)] = -1 + \frac{2H + F^2}{(\lambda-\lambda_1)(\lambda+\bar{\lambda}_1)} = -\frac{P(\lambda)}{(\lambda-\lambda_1)^2(\lambda+\bar{\lambda}_1)^2}$$

so that $P(\lambda)$ is constant in (x,t) and has roots at λ_1 and $-\bar{\lambda}_1$:

$$\begin{aligned} P(\lambda) &= (\lambda^2 + ic\lambda + b + \frac{1}{2}|u|^2)^2 - (u\lambda + icu + \frac{1}{2}u')(\bar{u}\lambda + ic\bar{u} - \frac{1}{2}\bar{u}') \\ &= \lambda^4 + 2ic\lambda^3 + (2b - c^2)\lambda^2 + 2i(a+bc)\lambda + b^2 - 2ac + 2d \\ &= (\lambda-\lambda_1)(\lambda+\bar{\lambda}_1)(\lambda-\lambda_2)(\lambda+\bar{\lambda}_2), \end{aligned}$$

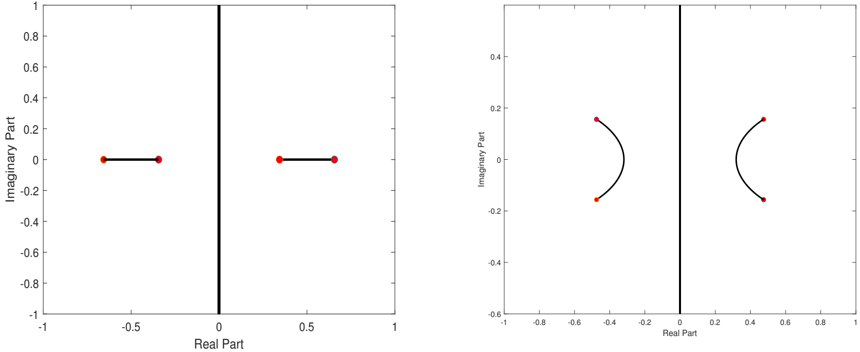
where constants (a,b,c,d) are incorporated from the second-order Lax-Novikov equation and its two conserved quantities.

Lax spectrum for the standing periodic waves

Two possible solutions for the standing periodic waves ($a = c = 0$):

$$u(x) = \mathrm{dn}(x; k), \quad u(x) = k\mathrm{cn}(x; k).$$

Solutions are periodic with some period and the Lax spectrum of $\varphi_x = U(\lambda, u)\varphi$ coincides with the Floquet spectrum.



Red dots show roots of $P(\lambda)$, e.g., eigenvalues of the nonlinearization method.

$u''' + 6|u|^2 u' + 2ic(u'' + 2|u|^2 u) + 4bu' + 8iau = 0$

Fix $\lambda = \lambda_1 \in \mathbb{C}$ with $\varphi = (p_1, q_1) \in \mathbb{C}^2$ and $\lambda = \lambda_2 \in \mathbb{C}$ with $\varphi = (p_2, q_2) \in \mathbb{C}^2$ such that $\lambda_1 \neq \pm\lambda_2$ and $\lambda_1 \neq \pm\bar{\lambda}_2$. Set

$$u = p_1^2 + \bar{q}_1^2 + p_2^2 + \bar{q}_2^2.$$

The spectral problem $\varphi_x = U(\lambda, u)\varphi$ becomes the Hamiltonian system with four conserved quantities:

$$\begin{aligned}
F_0 &= i\langle \mathrm{p}, \mathrm{q}\rangle, \\
F_1 &= \langle \Lambda \mathrm{p}, \mathrm{q}\rangle + \frac{1}{2}\langle \mathrm{p}, \mathrm{p}\rangle\langle \mathrm{q}, \mathrm{q}\rangle - \frac{1}{2}\langle \mathrm{p}, \mathrm{q}\rangle^2, \\
F_2 &= i\left[\langle \Lambda^2 \mathrm{p}, \mathrm{q}\rangle + \frac{1}{2}\langle \Lambda \mathrm{p}, \mathrm{p}\rangle\langle \mathrm{q}, \mathrm{q}\rangle + \frac{1}{2}\langle \mathrm{p}, \mathrm{p}\rangle\langle \Lambda \mathrm{q}, \mathrm{q}\rangle - \langle \mathrm{p}, \mathrm{q}\rangle\langle \Lambda \mathrm{p}, \mathrm{q}\rangle\right], \\
F_3 &= \langle \Lambda^3 \mathrm{p}, \mathrm{q}\rangle + \frac{1}{2}\langle \Lambda^2 \mathrm{p}, \mathrm{p}\rangle\langle \mathrm{q}, \mathrm{q}\rangle + \frac{1}{2}\langle \Lambda \mathrm{p}, \mathrm{p}\rangle\langle \Lambda \mathrm{q}, \mathrm{q}\rangle + \frac{1}{2}\langle \mathrm{p}, \mathrm{p}\rangle\langle \Lambda^2 \mathrm{q}, \mathrm{q}\rangle \\
&\quad - \frac{1}{2}\langle \Lambda \mathrm{p}, \mathrm{q}\rangle^2 - \langle \mathrm{p}, \mathrm{q}\rangle\langle \Lambda^2 \mathrm{p}, \mathrm{q}\rangle,
\end{aligned}$$

where $\mathrm{p} = (p_1, p_2, \bar{q}_1, \bar{q}_2)^t$, $\mathrm{q} = (q_1, q_2, -\bar{p}_1, -\bar{p}_2)^t$, $\Lambda := \mathrm{diag}(\lambda_1, \lambda_2, -\bar{\lambda}_1, -\bar{\lambda}_2)$.

Fourth-order Lax–Novikov equation

By differentiating of the constraints between u and (p,q), we obtain

$$\begin{aligned} u &= \langle \mathrm{p},\mathrm{p}\rangle, \\ u' + 2iF_0 u &= 2\langle \Lambda \mathrm{p},\mathrm{p}\rangle, \\ u'' + 2|u|^2 u + 2iF_0 u' - 4Hu &= 4\langle \Lambda^2 \mathrm{p},\mathrm{p}\rangle, \\ u''' + 6|u|^2 u' + 2iF_0(u'' + 2|u|^2 u) - 4Hu' + 8iKu &= 8\langle \Lambda^3 \mathrm{p},\mathrm{p}\rangle, \\ u'''' + 8|u|^2 u'' + 2u^2\bar{u}'' + 4u|u'|^2 + 6(u')^2\bar{u} + 6|u|^4 u & \\ +2iF_0(u''' + 6|u|^2 u') - 4H(u'' + 2|u|^2 u) + 8iKu' - 16Eu &= 16\langle \Lambda^4 \mathrm{p},\mathrm{p}\rangle, \end{aligned}$$

which yields the fourth-order Lax–Novikov equation:

$$\begin{aligned} & u'''' + 8|u|^2 u'' + 2u^2\bar{u}'' + 4u|u'|^2 + 6(u')^2\bar{u} + 6|u|^4 u \\ & \quad +2ic(u''' + 6|u|^2 u') + 4b(u'' + 2|u|^2 u) + 8iau' + 16du = 0, \end{aligned}$$

which is integrable with four conserved quantities. If u solves the second-order equation $u'' + 2|u|^2 u + 2icu' + 4bu = 0$, then the fourth-order equation is identically satisfied.

Integrability of the Hamiltonian system

The Hamiltonian system for (p_1, p_2, q_1, q_2) is obtained from the Lax equation

$$\frac{d}{dx}W(\lambda) = U(\lambda,u)W(\lambda) - W(\lambda)U(\lambda,u), \quad W(\lambda) = \begin{pmatrix} W_{11}(\lambda) & W_{12}(\lambda) \\ \bar{W}_{12}(-\lambda) & -\bar{W}_{11}(-\lambda) \end{pmatrix},$$

with

$$W_{11}(\lambda) = 1 - \sum_{j=1}^{2}\left(\frac{p_j q_j}{\lambda - \lambda_j} - \frac{\bar{p}_j \bar{q}_j}{\lambda + \bar{\lambda}_j}\right), \quad W_{12}(\lambda) = \sum_{j=1}^{2}\left(\frac{p_j^2}{\lambda - \lambda_j} + \frac{\bar{q}_j^2}{\lambda + \bar{\lambda}_j}\right).$$

Due to relations between u and squared eigenfunctions, we have

$$W_{11}(\lambda) = \frac{\lambda^4 + iT_1\lambda^3 + T_2\lambda^2 + iT_3\lambda + T_4}{(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda+\bar{\lambda}_1)(\lambda+\bar{\lambda}_2)}, \quad \left\{\begin{array}{l} T_1 = c, \\ T_2 = b + \frac{1}{2}|u|^2, \\ T_3 = a + \frac{1}{2}c|u|^2 - \frac{i}{4}(u'\bar{u} - u\bar{u}'), \\ T_4 = d + \frac{1}{2}b|u|^2 + \frac{i}{4}c(u'\bar{u} - u\bar{u}') + \frac{1}{8}(\end{array}\right.$$

$$W_{12}(\lambda) = \frac{S_0\lambda^3 + S_1\lambda^2 + S_2\lambda + S_3}{(\lambda-\lambda_1)(\lambda-\lambda_2)(\lambda+\bar{\lambda}_1)(\lambda+\bar{\lambda}_2)}, \quad \left\{\begin{array}{l} S_0 = u, \\ S_1 = \frac{1}{2}u' + icu, \\ S_2 = \frac{1}{4}(u'' + 2|u|^2 u) + \frac{i}{2}cu' + bu, \\ S_3 = \frac{1}{8}(u''' + 6|u|^2 u') + \frac{i}{4}c(u'' + 2|u|^2 \end{array}\right.$$

Third-order Lax–Novikov equation

If $T_4 = S_3 = 0$, then the fourth-order Lax–Novikov equation

$$u'''' + 8|u|^2 u'' + 2u^2 \bar{u}'' + 4u|u'|^2 + 6(u')^2 \bar{u} + 6|u|^4 u$$
$$+2ic(u''' + 6|u|^2 u') + 4b(u'' + 2|u|^2 u) + 8iau' + 16du = 0,$$

is satisfied by its reduction to the third-order Lax–Novikov equation

$$u''' + 6|u|^2 u' + 2ic(u'' + 2|u|^2 u) + 4bu' + 8iau = 0,$$

which is integrable with three conserved quantities:

$$d + \frac{1}{2} b|u|^2 + \frac{i}{4} c(u'\bar{u} - u\bar{u}') + \frac{1}{8}(u\bar{u}'' + u''\bar{u} - |u'|^2 + 3|u|^4) = 0,$$
$$2e - a|u|^2 - \frac{1}{4} c(|u'|^2 + |u|^4) + \frac{i}{8}(u''\bar{u}' - u'\bar{u}'') = 0,$$
$$f - \frac{i}{2} a(u'\bar{u} - u\bar{u}') + \frac{1}{4} b(|u'|^2 + |u|^4) + \frac{1}{16}(|u'' + 2|u|^2 u|^2 - (u'\bar{u} - u\bar{u}')^2) = 0.$$

Algebraic polynomial for the third-order equation

$\det W(\lambda)$ is constant in (x,t) and has simple poles at λ_1, λ_2, $-\bar{\lambda}_1$, and $-\bar{\lambda}_2$:

$$\det[W(\lambda)] = -\frac{\lambda^2 P(\lambda)}{(\lambda-\lambda_1)^2(\lambda-\lambda_2)^2(\lambda+\bar{\lambda}_1)^2(\lambda+\bar{\lambda}_2)^2},$$

with

$$P(\lambda) = \lambda^6 + 2ic\lambda^5 + (2b-c^2)\lambda^4 + 2i(a+bc)\lambda^3 + (b^2-2ac+2d)\lambda^2 + 2i(e+ab+cd)\lambda + f + 2bd - 2ce - a^2.$$

where constants (a,b,c,d,e,f) are incorporated from the third-order Lax-Novikov equation and its three conserved quantities.

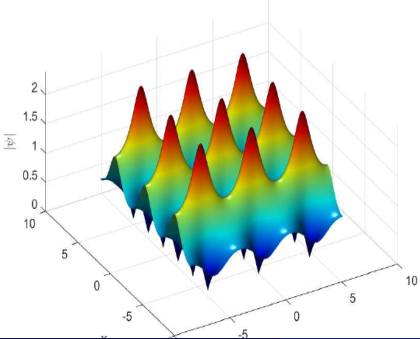
Lax spectrum for the double-periodic waves

Two possible solutions for the double-periodic waves ($a = c = e = 0$):

$$\psi(x,t) = k\frac{\mathrm{cn}(t;k)\mathrm{cn}(\sqrt{1+k}x;\kappa) + i\sqrt{1+k}\mathrm{sn}(t;k)\mathrm{dn}(\sqrt{1+k}x;\kappa)}{\sqrt{1+k}\mathrm{dn}(\sqrt{1+k}x;\kappa) - \mathrm{dn}(t;k)\mathrm{cn}(\sqrt{1+k}x;\kappa)}e^{it},$$

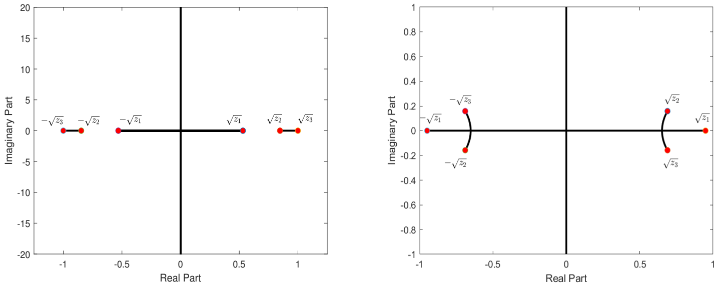
$$\psi(x,t) = \frac{\mathrm{dn}(t;k)\mathrm{cn}(\sqrt{2}x;\kappa) + i\sqrt{k(1+k)}\mathrm{sn}(t;k)}{\sqrt{1+k} - \sqrt{k}\mathrm{cn}(t;k)\mathrm{cn}(\sqrt{2}x;\kappa)}e^{ikt},$$

where $k \in (0,1)$ is elliptic modulus and $\kappa \in (0,1)$ is determined by k.



Lax spectrum for the double-periodic waves

Solutions are periodic in x with some period and the Lax spectrum of $\varphi_x = U(\lambda, u)\varphi$ coincides with the Floquet spectrum.



Red dots show roots of $P(\lambda)$, eigenvalues of the nonlinearization method.

Linearized NLS equation

Let ψ be a standing periodic wave solution of the NLS equation

$$i\psi_t + \frac{1}{2}\psi_{xx} + |\psi|^2\psi = 0.$$

Let χ be a perturbation of ψ. In the linearized approximation, it satisfies the linearized NLS equation

$$i\chi_t + \frac{1}{2}\chi_{xx} + 2|\psi|^2\chi + \psi^2\bar{\chi} = 0,$$

which is obtained from NLS after substituting $\psi + \chi$ to the NLS equation and neglecting χ^2, χ^3.

Spectral stability of standing waves

For the standing periodic waves, the variables can be separated:

$$\psi(x,t) = u(x+ct)e^{-2ibt}, \quad \chi(x,t) = v(x+ct)e^{-2ibt+\Lambda t},$$

where

$$\frac{1}{2}u'' + |u|^2 u + icu' + 2bu = 0$$

and

$$\Lambda v + \frac{1}{2}v'' + 2|u|^2 v + u^2 \bar{v} + icv' + 2bv = 0.$$

The spectral parameter Λ is found from the condition that $v(x)$ is bounded.

Since $u(x+L) = u(x)$ is periodic, then by Floquet theory, $v(x) = w(x)e^{i\theta x}$, where $\theta \in [-\pi/L, \pi/L]$ and $w(x+L) = w(x)$.

If there exists Λ with $\mathrm{Re}(\Lambda) > 0$ for some $\theta \in [-\pi/L, \pi/L]$, then the standing periodic wave is unstable in the time evolution of the NLS equation. It is modulationally unstable if the band with $\mathrm{Re}(\Lambda) > 0$ intersects $\Lambda = 0$ as $\theta \to 0$.

Relation to squared eigenfunctions

Recall the linear Lax system:

$$\varphi_x = U(\lambda,\psi)\varphi, \qquad U(\lambda,\psi) = \begin{pmatrix} \lambda & \psi \\ -\bar{\psi} & -\lambda \end{pmatrix}$$

and

$$\varphi_t = V(\lambda,\psi)\varphi, \qquad V(\lambda,\psi) = i \begin{pmatrix} \lambda^2 + \frac{1}{2}|\psi|^2 & \frac{1}{2}\psi_x + \lambda\psi \\ \frac{1}{2}\bar{\psi}_x - \lambda\bar{\psi} & -\lambda^2 - \frac{1}{2}|\psi|^2 \end{pmatrix},$$

where ψ is a solution of the NLS equation.

If φ and ϕ are two linearly independent solutions of the Lax system, then

Pair I	Pair II	Pair III
$\chi = \varphi_1^2 - \bar{\varphi}_2^2$	$\chi = \varphi_1\phi_1 - \bar{\varphi}_2\bar{\phi}_2$	$\chi = \phi_1^2 - \bar{\phi}_2^2$
$\chi = i\varphi_1^2 + i\bar{\varphi}_2^2$	$\chi = i\varphi_1\phi_1 + i\bar{\varphi}_2\bar{\phi}_2$	$\chi = i\phi_1^2 + i\bar{\phi}_2^2$

are solutions of the linearized NLS equation.

Relation to squared eigenfunctions

Theorem

Let λ belongs to the Lax spectrum so that

$$\varphi(x,t) = \xi(x+ct)e^{-2ib\sigma_3 t + \Omega t}$$

with $\xi \in L^\infty(\mathbb{R})$. Then, $\Omega = \pm i\sqrt{P(\lambda)}$, where $P(\lambda)$ is the polynomial for the second-order Lax–Novikov equation:

$$P(\lambda) = \lambda^4 + 2ic\lambda^3 + (2b - c^2)\lambda^2 + 2i(a+bc)\lambda + b^2 - 2ac + 2d$$

Consequently, $\Lambda = 2\Omega = \pm 2i\sqrt{P(\lambda)}$.

The proof follows from separation of variables for

$$\xi_x = U(\lambda,u)\xi, \qquad U(\lambda,u) = \begin{pmatrix} \lambda & u \\ -\bar{u} & -\lambda \end{pmatrix}$$

$$\Omega\xi + c\xi_x - 2ib\sigma_3\xi = V(\lambda,u)\xi, \quad V(\lambda,u) = i\begin{pmatrix} \lambda^2 + \frac{1}{2}|u|^2 & \frac{1}{2}u_x + \lambda u \\ \frac{1}{2}\bar{u}_x - \lambda\bar{u} & -\lambda^2 - \frac{1}{2}|u|^2 \end{pmatrix},$$

Instability of the dnoidal periodic waves

$$u(x) = \mathrm{dn}(x; k), \quad L = 2K(k).$$

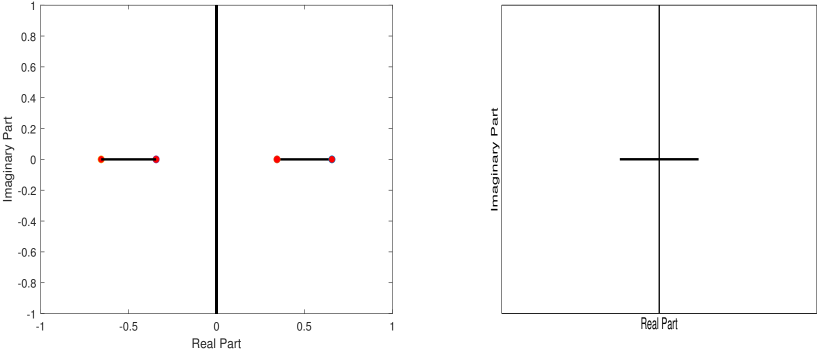


Figure: Left: Lax spectrum. Right: stability spectrum related by $\Lambda = \pm 2i\sqrt{P(\lambda)}$.

Instability of the cnoidal periodic waves

$$u(x) = k\,\mathrm{cn}(x;k), \quad L = 4K(k).$$

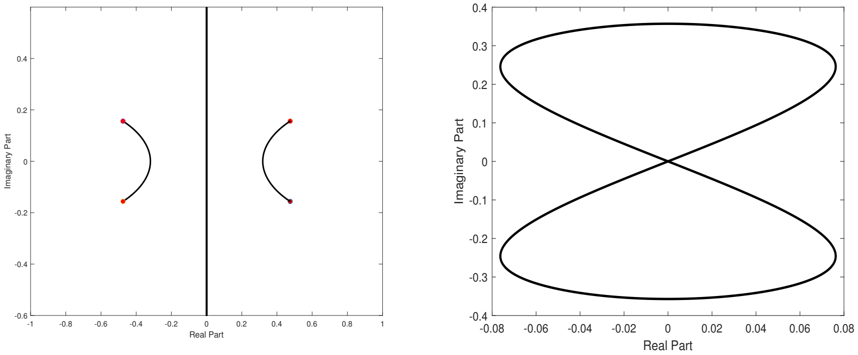


Figure: Left: Lax spectrum. Right: stability spectrum related by $\Lambda = \pm 2i\sqrt{P(\lambda)}$.

Spectral stability of double-periodic waves

For the double-periodic waves, the variables can not be separated:

$$\psi(x,t) = [q(x,t) + i\delta(t)]e^{it+i\alpha(t)},$$

where $q(x+L,t) = q(x,t+T) = q(x,t)$, $\delta(t+T) = \delta(t)$, $\alpha(t+T) = \alpha(t)$.
Perturbation $\chi(x,t)$ to $\psi(x,t)$ satisfies the linearized NLS equation

$$i\chi_t + \frac{1}{2}\chi_{xx} + 2|\psi|^2\chi + \psi^2\bar{\chi} = 0,$$

Due to periodicity, we can think of the Floquet theory both with respect to x and t to represent the perturbation in the form

$$\chi(x,t) = v(x,t)e^{it+i\theta x+\Lambda t},$$

where $v(x+L,t) = v(x,t+T) = v(x,t)$, $\theta \in [-\pi/L, \pi/L]$, and Λ is somehow defined (unique if $\mathrm{Im}(\Lambda) \in [-\pi/T, \pi/T]$).

Spectral stability of double-periodic waves

Recall the linear Lax system

$$\varphi_x = U(\lambda,\psi)\varphi, \qquad U(\lambda,\psi) = \begin{pmatrix} \lambda & \psi \\ -\bar{\psi} & -\lambda \end{pmatrix}$$

and

$$\varphi_t = V(\lambda,\psi)\varphi, \qquad V(\lambda,\psi) = i\begin{pmatrix} \lambda^2 + \frac{1}{2}|\psi|^2 & \frac{1}{2}\psi_x + \lambda\psi \\ \frac{1}{2}\bar{\psi}_x - \lambda\bar{\psi} & -\lambda^2 - \frac{1}{2}|\psi|^2 \end{pmatrix},$$

where ψ is a solution of the NLS equation.

By the Floquet theory both with respect to x and t, we write

$$\varphi(x,t) = \xi(x,t)e^{i\theta x + t\Omega},$$

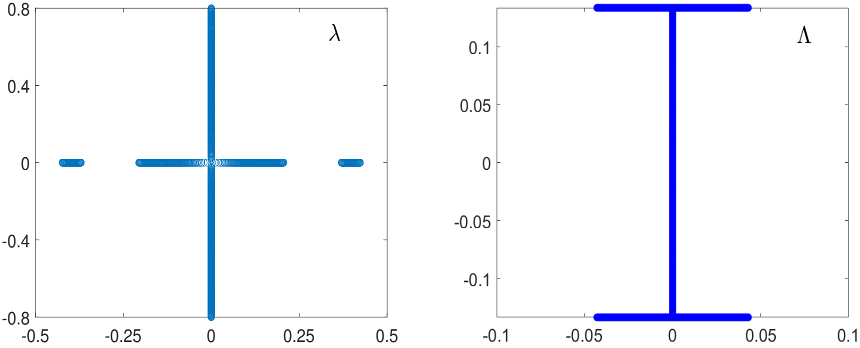
$\xi(x+L,t) = \xi(x,t+T) = \xi(x,t)$, $\theta \in [-\pi/L, \pi/L]$, $\mathrm{Im}(\Omega) \in [-\pi/T, \pi/T]$.

- λ is found from the Lax spectrum for $\varphi_x = U(\lambda,\psi)$.
- Ω is found from $\varphi_t = V(\lambda,\psi)\varphi$.

Is there a relation between Ω and $P(\lambda)$ for the double-periodic solution ψ?

Instabilities of the first solution

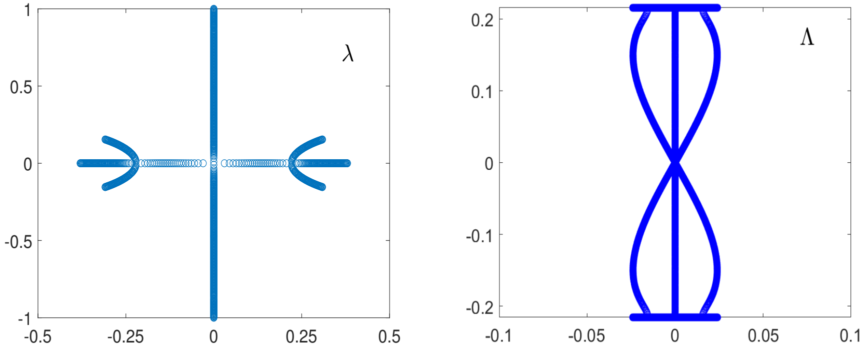
$k = 0.85$ (Pelinovsky, 2021):



Left: Lax spectrum. Right: stability spectrum.

Instabilities of the second solution

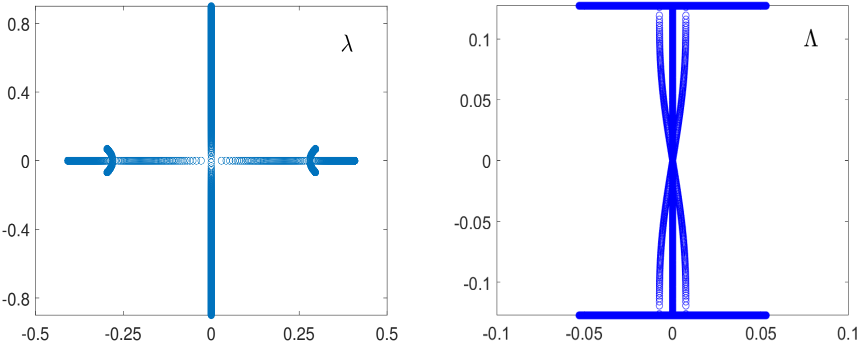
$k = 0.6$ (Pelinovsky, 2021):



Left: Lax spectrum. Right: stability spectrum.

Instabilities of the second solution

$k = 0.9$ (Pelinovsky, 2021):

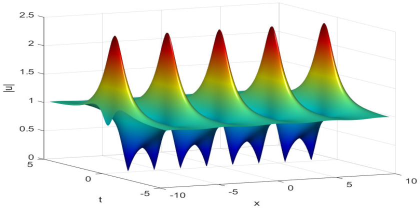


Left: Lax spectrum. Right: stability spectrum.

Akhmediev breathers

In the limit $k \to 1$ both families converge to a particular example of the Akhmediev breather (AB):

$$\psi(x,t) = \frac{\cos(\sqrt{2}x) + i\sqrt{2}\sinh(t)}{\sqrt{2}\cosh(t) - \cos(\sqrt{2}t)} e^{it}.$$

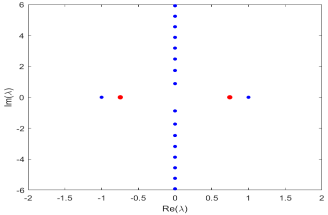


Akhmediev breathers under periodic perturbation

A family of Akhmediev breathers with parameter $\lambda \in (0,1)$:

$$\psi(x,t) = e^{it}\left[1 - \frac{2(1-\lambda^2)\cosh(k\lambda t) + ik\lambda\sinh(k\lambda t)}{\cosh(k\lambda t) - \lambda\cos(kx)}\right],$$

If the perturbation is periodic, the Lax and stability spectra are purely discrete.
There was an open question if the Akhmediev breather is linearly unstable.
P. Grinevich & P. Santini, Nonlinearity **34** (2021) 8331–8358
M. Haragus & D. Pelinovsky, J. Nonlinear Science **32** (2022) 66



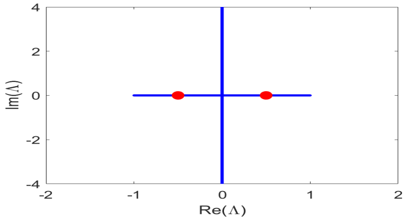


Figure: Lax spectrum (left) and stability spectrum (right) of Akhmediev breather.

Other examples of integrable Hamiltonian systems

- Modified Korteweg–de Vries equation

$$u_t + 6u^2 u_x + u_{xxx} = 0$$

Dnoidal periodic waves are modulationally stable.
Cnoidal periodic waves are modulationally unstable.
J. Chen & D. Pelinovsky, Nonlinearity **31** (2018) 1955–1980

- Sine–Gordon equation

$$u_{tt} - u_{xx} + \sin(u) = 0$$

Same conclusion.
D. Pelinovsky & R. White, Proceedings A **476** (2020) 20200490

- Derivative NLS equation

$$i\psi_t + \psi_{xx} + i(|\psi|^2\psi)_x = 0.$$

There exist modulationally stable periodic waves.
J. Chen, D. Pelinovsky, & J. Upsal, J. Nonlinear Science **31** (2021) 58

Summary

- Standing periodic waves are solutions of the second-order Lax–Novikov equation. Double-periodic waves are solutions of the third-order Lax–Novikov equation. Akhmediev and Kuznetsov–Ma breathers are particular cases of double-periodic solutions.
- Standing periodic waves are spectrally (modulationally) unstable, their instability is computed from separation of variables and Floquet theory.
- Double-periodic waves are also linearly unstable, their instability is computed from double Floquet theory (both in x and t).
- Breathers are also linearly unstable.

Many thanks for your attention!