# Internal modes of discrete solitons near the anti-continuum limit of the dNLS equation

Dmitry Pelinovsky, Anton Sakovich (McMaster University)

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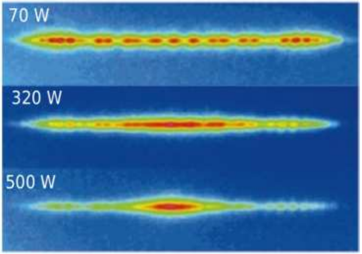
# dNLS and Discrete Optical Solitons




The dNLS equation

$$i\frac{dE_n}{dz} + \beta E_n + C\left(E_{n-1} + E_{n+1}\right) + \gamma |E_n|^2 E_n = 0,$$

is used to model propagation of light in a coupled array of optical waveguides.

Eisenberg et al. *PRL* 81, 3383 (1998)
Christodoulides et al. *Nature* 424, 817 (2003)

# Anti-continuum limit for the dNLS equation

Consider the discrete nonlinear Schrödinger (dNLS) equation

$$\boxed{i\dot{u}_n + \epsilon\left(u_{n-1} - 2u_n + u_{n+1}\right) + |u_n|^{2p} u_n = 0,}$$

where $\mathbf{u}(\cdot) : \mathbb{R} \to \mathbb{C}^{\mathbb{Z}}$, $p \geq 1$ is an integer, and $\epsilon \in \mathbb{R}$ is a coupling constant.

Thanks to a numerical approximation to the second-order derivative

$$f_n'' = \frac{f_{n-1} - 2f_n + f_{n+1}}{h^2} + \mathcal{O}(h^2),$$

one can think that $\boxed{\epsilon \sim h^{-2}}$, where $h$ is the lattice spacing.

The *anti-continuum limit* is the limit of $\epsilon \to 0$ ($h \to \infty$).

# Discrete solitons

We define *discrete solitons* of the dNLS as real stationary envelope $\phi$ in the time-periodic solutions

$$u_n(t) = \phi_n e^{it},$$

so that

$$\phi_n \left(1 - \phi_n^{2p}\right) = \epsilon(\phi_{n+1} - 2\phi_n + \phi_{n-1}).$$

In the anti-continuum limit, $\epsilon \to 0$, we simply get

$$\phi_n^{(0)} \left(1 - \left[\phi_n^{(0)}\right]^{2p}\right) = 0 \qquad \boxed{\phi_n^{(0)} \in \{-1, 0, 1\}},$$

so it is natural to expand discrete solitons in powers of the coupling parameter,

$$\phi_n = \phi_n^{(0)} + \epsilon\phi_n^{(1)} + \epsilon^2\phi_n^{(2)} + \dots .$$

# Properties of discrete solitons

Suppose in the anti-continuum limit discrete solitons are supported on a set

$$U = \left\{ n \in \mathbb{Z} : \ \phi_n^{(0)} \neq 0 \right\},$$

where $N = |U|$ is the number of active nodes in the anti-continuum limit.

**Theorem (MacKay, Aubry 94)**

*Suppose $|U| < \infty$, then for each $\phi^{(0)}$ and sufficiently small $\epsilon$, there are positive constants $C$ and $\kappa$ and a unique solution $\phi \in l^2(\mathbb{Z})$ such that*

$$\left\| \phi - \phi^{(0)} \right\|_{l^2} \leq C\,|\epsilon| \quad \text{and} \quad |\phi_n| \leq C e^{-\kappa |n|}.$$

# Spectral stability of discrete solitons

We consider a small perturbation of a discrete soliton

$$u_n(t) = e^{it}\left[\phi_n + (v_n + iw_n)\,e^{\lambda t} + (\bar{v}_n + i\bar{w}_n)\,e^{\bar{\lambda} t}\right],$$

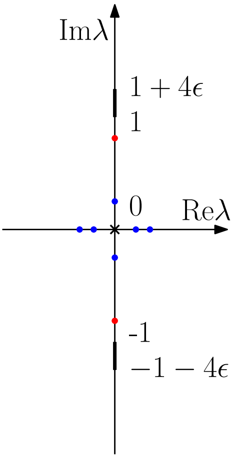
with $\mathbf{v}, \mathbf{w} \in \mathbb{C}^{\mathbb{Z}}$ and $\lambda \in \mathbb{C}$ spectral parameter. Linearized equations give the eigenvalue problem,

$$\boxed{L_+\mathbf{v} = -\lambda \mathbf{w}, \qquad L_-\mathbf{w} = \lambda \mathbf{v},}$$

where

$$\begin{aligned}
(L_+\mathbf{v})_n &= \left(-\epsilon\Delta + 1 - (2p+1)\phi_n^{2p}\right) v_n, \\
(L_-\mathbf{w})_n &= \left(-\epsilon\Delta + 1 - \phi_n^{2p}\right) w_n.
\end{aligned}$$

# Spectrum near the anti-continuum limit

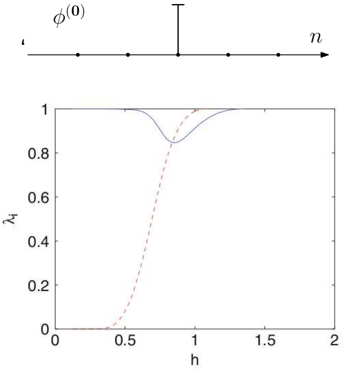



Let $n_0$ be the number of sign differences in the compact configuration $\phi^{(0)}$.
For small $\epsilon > 0$ there are

- two bands of continuous spectrum $i[-1-4\epsilon, -1]$ and $i[1, 1+4\epsilon]$
- $N-1-n_0$ pairs of real eigenvalues and $n_0$ pairs of pure imaginary eigenvalues[a]
- there could also be internal modes bifurcating from continuous bands

[a] Pelinovsky *et al.*, *Physica D* 212, 1–19 (2005)

# Example $N = 1$

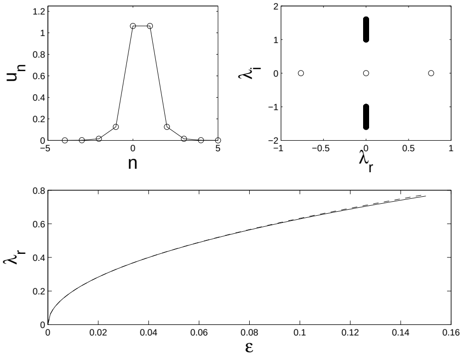



Bifurcation of internal modes from the continuous spectrum for the single-site discrete soliton $N = 1$ for the cubic dNLS equation $(p = 1)$. Since $n_0 = 0$ and $N - 1 - n_0 = 0$ there are no eigenvalues bifurcating from zero

The figure is taken from P.G. Kevrekidis, *The Discrete Nonlinear Schrödinger Equation*, Springer Tracts in Modern Physics, vol. 232, Springer, NY, 2009.

# Example $N = 2$

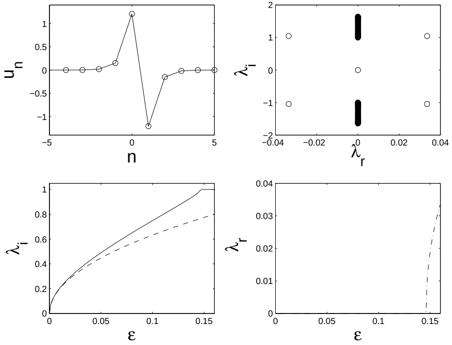
Bifurcation from the zero eigenvalue for a two-site discrete soliton with $\phi_n^{(0)} = \delta_{n,0} + \delta_{n,1}$. Since $n_0 = 0$ and $N - 1 - n_0 = 1$, one pair of unstable eigenvalues bifurcates along the real axis.




The figure is taken from Pelinovsky *et al.*, *Physica D* 212, 1–19 (2005).

# Example $N = 2$

Bifurcation from the zero eigenvalue for a two-site discrete soliton with $\phi_n^{(0)} = \delta_{n,0} - \delta_{n,1}$. Since $n_0 = 1$ and $N - 1 - n_0 = 0$,
one pair of stable eigenvalues bifurcates along the imaginary axis.




The figure is taken from Pelinovsky *et al.*, *Physica D* 212, 1–19 (2005).

# Goal

We would like to prove that near the anti-continuum limit the spectrum of the linearized operator has no internal modes (isolated purely imaginary eigenvalues) near the continuous spectral bands.

We analyze the *resolvent* of the linearized operator using the fact that the discrete soliton $\phi$ is compactly supported as $\epsilon \to 0$.

# The resolvent operator

Consider the eigenvalue problem

$$\begin{aligned} L_+ \mathbf{v} + \lambda \mathbf{w} &= 0, \\ L_- \mathbf{w} - \lambda \mathbf{v} &= 0. \end{aligned}$$

The resolvent $\boxed{R(\lambda) = (\mathcal{L} - \lambda I)^{-1}}$ is a solution operator to the corresponding inhomogeneous problem

$$\begin{aligned} -\epsilon (\Delta \mathbf{v})_n + v_n - (2p+1)\phi_n^{2p} v_n + \lambda w_n &= F_n, \\ -\epsilon (\Delta \mathbf{w})_n + w_n - \phi_n^{2p} w_n - \lambda v_n &= G_n, \end{aligned}$$

where $\mathbf{F}, \mathbf{G} \in l^2$ and

$$\phi_n^{2p} = \sum_{m \in U} \delta_{n,m}(1 + \epsilon \chi_m) + \epsilon^2 W_n,$$

where $\{\chi_m\}_{m \in U}$ are some numerical coefficients and $\mathbf{W} \in l^2(\mathbb{Z})$ is bounded potential as $\epsilon \to 0$.

# Resolvent for the discrete Laplacian

The resolvent for the discrete Laplacian $R_0(\lambda) = (\Delta - \lambda I)^{-1}$ can be written explicitly as

$$\boxed{(R_0(\lambda)\mathbf{f})_n = \frac{1}{2i \sin z(\lambda)} \sum_m f_m e^{-iz(\lambda)|n-m|}},$$

where $z(\lambda)$, $\lambda \in \mathbb{C}$ is a unique solution of the transcendental equation

$$2(1 - \cos z(\lambda)) = \lambda, \quad \mathrm{Re} z(\lambda) \in [-\pi, \pi), \qquad \mathrm{Im} z(\lambda) \leq 0.$$

If $\lambda \notin \sigma(-\Delta) = [0, 4]$, then $\mathrm{Im} z(\lambda) < 0$, and

$$R_0(\lambda) : l^2 \mapsto l^2, \qquad \lambda \notin \sigma(-\Delta) = [0, 4].$$

# Resolvent for the discrete Laplacian

For the continuous spectrum of $\Delta$ we define resolvent as a limit

$$R_0^{\pm}(\omega) = \lim_{\epsilon \downarrow 0} R_0(\omega \pm i\epsilon).$$

Since

$$(R_0^{\pm}(\omega)\mathbf{f})_n = \frac{1}{2i\sin\theta(\omega)} \sum_m f_m e^{\mp i\theta(\omega)|n-m|}, \qquad 2(1-\cos\theta(\omega)) = \omega \in [0,4].$$

we have

$$\left\| R_0^{\pm}(\omega)\mathbf{f} \right\|_{l^\infty} \leq \frac{1}{2\sin\theta(\omega)} \|\mathbf{f}\|_{l^1}.$$

Therefore,

$$R_0^{\pm}(\omega) : l^1(\mathbb{Z}) \mapsto l^\infty(\mathbb{Z}), \qquad \omega \in (0,4)$$

and resonance occurs as $\omega \to 0$ and $\omega \to 4$.

# Boundedness of the full resolvent

The following theorem formulated for $\boxed{\lambda = i\Omega}$ rules out existence of discrete spectrum in a small neighborhood of the continuous bands.

## Theorem

*Suppose the set of active nodes $U$ has no holes and $p \geq 2$. There are $\epsilon_0 > 0$ and $\delta > 0$ such that for any fixed $\epsilon \in (0, \epsilon_0)$ the resolvent operator*

$$R(\Omega) : l^2 \times l^2 \mapsto l^2 \times l^2$$

*is bounded for any $\Omega \notin [-1 - 4\epsilon, -1] \cup B_\delta(0) \cup [1, 1 + 4\epsilon]$. Moreover, $R(\Omega)$ has exactly $2N$ poles (with the account of their multiplicities) inside $B_\delta(0)$ and admits the uniformly continuous limits*

$$R^{\pm}(\Omega) = \lim_{\mu \downarrow 0} R(\Omega \pm i\mu) : l^1_1 \times l^1_1 \mapsto l^\infty \times l^\infty,$$

*for any $\Omega \in [1, 1 + 4\epsilon] \cup [-1 - 4\epsilon, -1]$.*

# Exceptions from the theorem

The same theorem hods for $p = 1$ if $N = 1$ (fundamental soliton).

If $p = 1$ and $N \geq 2$, degeneracy of the resolvent operator near the end points of the continuous spectrum at $\Omega = \pm 1$ and $\Omega = \pm(1 + 4\epsilon)$ (resonances) requires computations of perturbation theory beyond the first order.

If the set of active nodes $U$ has some holes, the limiting resolvent operator is singular in the interior points of the continuous spectrum (resonant poles). Boundness of the resolvent operator can only be clarified with higher-order perturbation theory.

# Two-site soliton without holes $N = 2$

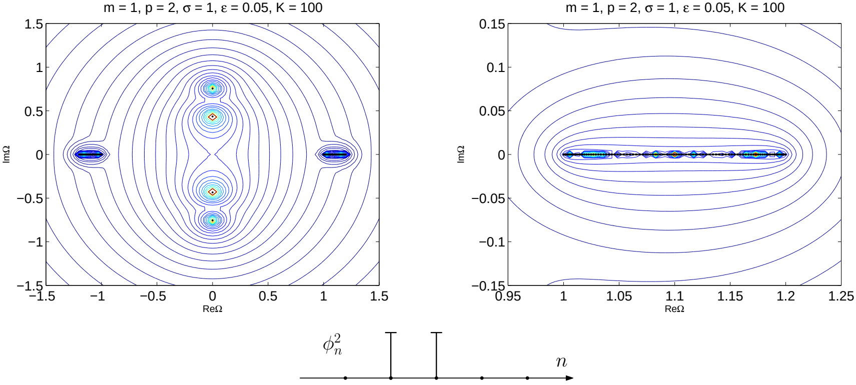



Figure: Level sets for $\left\|(L - \Omega I)^{-1}\right\|_2$ in $\Omega$-plane for a discrete soliton $\phi$ supported on $U = \{0, 1\}$. Here $L$ is a $(4K+2) \times (4K+2)$ matrix approximation of operator $\mathcal{L}$.

# Two-site soliton with a hole $N = 2$

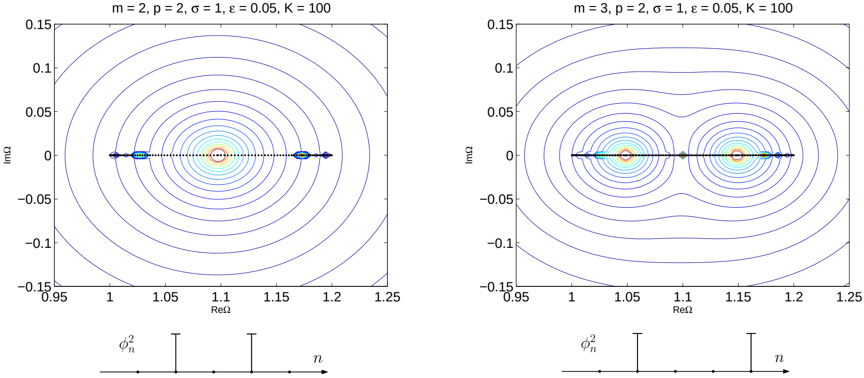



Figure: Level sets for $\left\|(L - \Omega I)^{-1}\right\|_2$ in $\Omega$-plane for a discrete soliton $\phi$ supported on $U = \{0, 2\}$ (Left) and on $U = \{0, 3\}$ (Right).

# Discussion: asymptotic stability of discrete solitons

We would like to consider asymptotic stability of discrete solitons

$$\boxed{i\dot{u}_n + \epsilon\left(u_{n-1} - 2u_n + u_{n+1}\right) - V_n u_n + \left|u_n\right|^{2p} u_n = 0,}$$

where $V : \mathbb{Z} \to \mathbb{R}$ is a trapping potential.

Assume that the discrete solitons $u_n(t) = \phi_n e^{i\omega t}$ exist for some $\omega \in \mathbb{R}$ and are orbitally stable in $l^2(\mathbb{R})$, that is, for any $\epsilon > 0$ there is a $\delta(\epsilon) > 0$, such that if $\|u(0) - \phi\|_{l^2} \leq \delta(\epsilon)$ then

$$\inf_{\theta\in\mathbb{R}} \|u(t) - e^{i\theta}\phi\|_{l^2} \leq \epsilon,$$

for all $t > 0$.

- Kevrekidis, Pelinovsky, Stefanov (2009) and Cuccagna, Tarulli (2009) proved asymptotic stability of discrete solitons for $p \geq 3$ using Stritcharz analysis.
- Pelinovsky, Mizumachi (2011) improved the results with pointwise decay estimates for $p > 2.75$.
- No results are available for $p = 1$ and $p = 2$.