Solitons and breathers on wave background

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SouthEast University, Nanjing, China, August 21, 2023

Background of the problem

We are dealing with the canonical model for the shallow water waves, the Korteweg–de Vries (KdV) equation:

$$u_t + 6uu_x + u_{xxx} = 0, \qquad \text{(KdV)}$$

with the step-like data

$$\lim_{x\to-\infty} u(t,x) = u_-, \qquad \lim_{x\to+\infty} u(t,x) = u_+.$$

Applications: tidal bores, earthquake-generated waves

G. A. El, Adv. Fluid Mech. **47** (2007) 19-53

G. A. El and M. A. Hoefer, Physica D **333** (2016) 11-65

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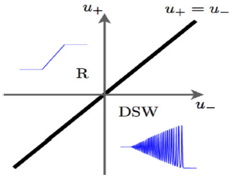
The step-like initial data results in the appearance of a rarefaction wave (RW) if $u_+ > u_-$ and a dispersive shock wave (DSW) if $u_+ < u_-$.



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Soliton propagation on RW and DSW background have been considered recently:

M. D. Maiden, D. V. Anderson, A. A. Franco, G. A. El, and M. A. Hoefer, Phys. Rev. Lett. **120** (2018) 144101

P. Sprenger, M. A. Hoefer, and G. A. El, Phys. Rev. E **97** (2018) 032218

T. Congy, G. A. El and M. A. Hoefer, J. Fluid Mech. **875** (2019) 1145–1174

K. van der Sande, G. A. El and M. A. Hoefer, J. Fluid Mech. **928** (2021) A21

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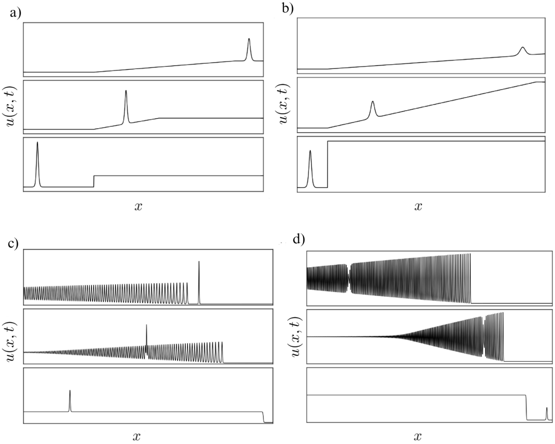
This is the toy model for soliton gases analyzed in mathematics and observed in hydrodynamical experiments:

M. Bertola, R. Jenkins, and A. Tovbis (2022), arXiv: 2210.01350
M. Girotti, T. Grava, R. Jenkins, K. McLaughlin A. Minakov (2022), arXiv:2205.02601
T. Congy, G. A. El, G. Roberti, and A. Tovbis (2022) arXiv:2208.04472
Y. Mao, S. Chandramouli, W. Xu, and M. A. Hoefer (2023) arXiv: 2302.11161

State of the art



Soliton-RW:
a) Tunneling.
b) Trapping.

Soliton-DSW:
a) Tunneling.
b) Trapping.

M. J. Ablowitz, J. T. Cole, M. A. Hoefer, Stud. Appl. Math. (2023) in print.

State of the art

Soliton-RW: Depending on the initial amplitude of a solitary wave, it is either transmitted over or trapped inside the RW background.

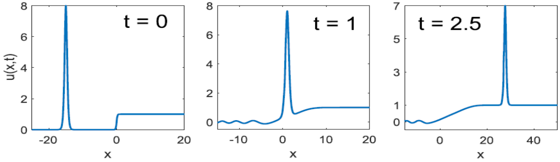


FIG. 1. Tunneling soliton example for $\kappa_1 = 2$, $c = 1$, $x_0 = -15$.

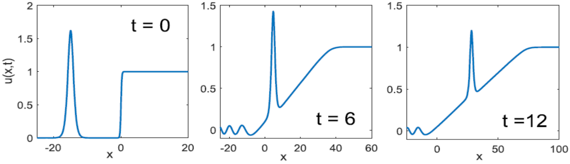


FIG. 2. Trapped soliton example for $\kappa_0 = 0.9$, $c = 1$, $x_0 = -15$.

M. J. Ablowitz, X. D. Luo, and J. T. Cole, J. Math. Phys. **59** (2018), 091406

State of the art

This phenomenon was interpreted from the inverse scattering method:

$$\mathcal{L}v = \lambda v, \qquad \mathcal{L} := -\frac{\partial^2}{\partial x^2} - u$$

and

$$\frac{\partial v}{\partial t} = \mathcal{M}v, \qquad \mathcal{M} := -3u_x - 6u\frac{\partial}{\partial x} - 4\frac{\partial^3}{\partial x^3},$$

where $\lim_{x\to-\infty} u(x,t) = 0$ and $\lim_{x\to+\infty} u(x,t) = c^2$.

- Transmitted soliton corresponds to an isolated eigenvalue of $\mathcal{L}$ in $(-\infty, -c^2)$ outside the continuous spectrum on $[-c^2, \infty)$.
- Trapped soliton corresponds to a “pseudo–embedded" eigenvalue inside the continuous spectrum of $\mathcal{L}$ in $[-c^2, 0]$

M. J. Ablowitz, X. D. Luo, and J. T. Cole, J. Math. Phys. **59** (2018), 091406

State of the art

Soliton-DSW: The rigorous IST method was applied for the step-like boundary conditions:

- N solitons added as poles in the IST method scatter towards zero boundary conditions as t evolves.
- Phase shifts of the N solitons were appropriately computed.
- These N solitons are considered to be transmitted solitons over the DSW background.
- No differences between transmitted and trapped solitons appear in the IST method.

I. Egorova, Z. Gladka, V. Kotlyarov, and G. Teschl, Nonlinearity **26** (2013) 1839

I. Egorova, J. Michor, and G. Teschl, arXiv: 2109.08423 (2021)

Soliton-RW interaction: transmitted soliton

Main result: Transmitted soliton on the RW background can be constructed via Darboux transformation.

A. Mucalica and D.E. Pelinovsky, Solitons on the rarefaction wave background via the Darboux transformation, Proc. R. Soc. A 478 (2022) 20220474

Darboux transformation: Let u be a solution of the KdV equation and v_0 be a real solution of the Lax equations for $\lambda = \lambda_0 \in \mathbb{R}$ such that $v_0 \neq 0$. Then,

$$\hat{u} := u + 2\frac{\partial^2}{\partial x^2}\log(v_0)$$

is a new solution of the KdV equation.

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is a new solution of the KdV equation.

If λ_0 is below the bottom of the spectrum of $\mathcal{L} = -\partial_x^2 - u$, then $v_0 \neq 0$ everywhere by Sturm's nodal theory.

Soliton-RW interaction: transmitted soliton

Let $u(x) = 0$ for $x < 0$ and $u(x) = c^2$ for $x > 0$ at initial time $t = 0$.
Pick $\lambda_0 = -\mu_0^2 < -c^2$ and obtain

$$v_0(x) = \begin{cases} e^{\mu_0(x-x_0)} + e^{-\mu_0(x-x_0)}, & x < 0, \\ c_1 e^{\nu_0 x} + c_2 e^{-\nu_0 x}, & x > 0, \end{cases}$$

where $\nu_0 := \sqrt{\mu_0^2 - c^2} > 0$, x_0 is arbitrary, and (c_1, c_2) are uniquely found from the continuity of v_0 and v_0' across $x = 0$.

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where $\nu_0 := \sqrt{\mu_0^2 - c^2} > 0$, x_0 is arbitrary, and (c_1, c_2) are uniquely found from the continuity of v_0 and v_0' across $x = 0$.

The new solution is given by

$$\hat{u}(x) = \begin{cases} 2\mu_0^2 \operatorname{sech}^2[\mu_0(x - x_0)], & x < 0, \\ c^2 + 4\nu_0^2 \frac{\nu_0^2 + \mu_0^2 + (\nu_0^2 - \mu_0^2)\cosh(2\mu_0 x_0)}{[(\nu_0+\mu_0)\cosh(\nu_0 x - \mu_0 x_0) + (\nu_0 - \mu_0)\cosh(\nu_0 x + \mu_0 x_0)]^2}, & x > 0. \end{cases}$$

The solution is bounded if $x_0 \leq 0$.

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where $\nu_0 := \sqrt{\mu_0^2 - c^2} > 0$, x_0 is arbitrary, and (c_1, c_2) are uniquely found from the continuity of v_0 and v_0' across $x = 0$.

- The solitary wave decays differently as $x \to -\infty$ (decay rate is μ_0) and as $x \to +\infty$ (decay rate is $\nu_0 = \sqrt{\mu_0^2 - c^2}$).
- The Lax spectrum of the new solution is $[-c^2, \infty)$ and a simple isolated eigenvalue $\lambda_0 = -\mu_0^2 < -c^2$.

Soliton-RW interaction: trapped soliton

Some evidences that no trapped solitons actually exist.

- Eigenfunctions of $\mathcal{L}v = \lambda_0 v$ are bounded but not decaying if $\lambda \in [-c_0^2, \infty)$. No embedded eigenvalues exist if

$$u(x) \to c^2 \quad \text{as} \;\; x \to +\infty \qquad \text{rapidly}$$

- Darboux transformation does not produce any bounded solutions if $\lambda_0 \in [-c_0^2, \infty)$.

We shall prove that no trapped soliton exists for one example.

A. Mucalica and D.E. Pelinovsky, Solitons on the rarefaction wave background via the Darboux transformation, Proc. R. Soc. A 478 (2022) 20220474

Soliton-RW interaction: trapped soliton

Consider a linear superposition of a soliton and the step function:

$$u(x) = 2\mu_0^2 \text{sech}^2(\mu_0(x - x_0)) + c^2 H(x),$$

where $\mu_0 > 0$, $x_0 < 0$, and $H(x)$ is the Heaviside step function.

Eigenfunctions of $\mathcal{L}v = \lambda v$ with $\lambda = k^2$ are known explicitly:

$$\phi(x;k) = e^{-ikx}\left[1 - \frac{i\mu_0}{k + i\mu_0} e^{\mu_0(x-x_0)} \text{sech}(\mu_0(x - x_0))\right], \quad x < 0$$

and

$$\psi(x;k) = e^{i\varkappa x}\left[1 - \frac{i\mu_0}{\varkappa + i\mu_0} e^{-\mu_0(x-x_0)} \text{sech}(\mu_0(x - x_0))\right], \quad x > 0,$$

where $\varkappa := \sqrt{c^2 + k^2}$.

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The scattering data are obtained from the scattering relation:

$$\phi(x;k) = a(k)\overline{\psi}(x;k) + b(k)\psi(x;k), \quad x \in \mathbb{R},$$

where $\bar{\psi}$ is obtained from ψ by reflection $\varkappa \mapsto -\varkappa$.

Straightforward computation yields:

$$a(k) = \frac{(\varkappa + k)\left(\varkappa k + \mu_0^2 + i\mu_0(\varkappa - k)\tanh(\mu_0 x_0)\right)}{2\varkappa(\varkappa + i\mu_0)(k + i\mu_0)}.$$

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where $\mu_0 > 0$, $x_0 < 0$, and $H(x)$ is the Heaviside step function.

$\phi(x;k)$ and $a(k)$ can be continued analytically for $k \in \mathbb{C}$, $\text{Im}(k) > 0$. Zeros of $a(k)$ for $\text{Im}(k) > 0$ correspond to solutions of $\mathcal{L}v = \lambda v$ with $\lambda = k^2$ and $\phi(x;k) \to 0$ as $x \to -\infty$.

However, $k = ic$ is a branch point for $\kappa := \sqrt{c^2 + k^2}$. Branch cuts must be defined on $i\mathbb{R}$.

Soliton-RW interaction: trapped soliton

Consider a linear superposition of a soliton and the step function:

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By using

$$a(k) = \frac{(\varkappa + k)\left(\varkappa k + \mu_0^2 + i\mu_0(\varkappa - k)\tanh(\mu_0 x_0)\right)}{2\varkappa(\varkappa + i\mu_0)(k + i\mu_0)},$$

we are looking for roots $k \in \mathbb{C}$ with $\text{Im}(k) > 0$ of equation

$$\varkappa k + \mu_0^2 + i\mu_0(\varkappa - k)\tanh(\mu_0 x_0) = 0.$$

If $x_0 \to -\infty$, $k = i\mu_0$ is a simple root of this equation for the eigenvalue $\lambda_0 = k^2 = -\mu_0^2$ with either $\mu_0 \in (c, \infty)$ or $\mu_0 \in (0, c)$.

Soliton-RW interaction: trapped soliton

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where $\mu_0 > 0$, $x_0 < 0$, and $H(x)$ is the Heaviside step function.

For every sufficiently large $x_0 \ll -1$, there exists an isolated real eigenvalue $\lambda_0 \approx -\mu_0^2$ if $\mu_0 \in (c, \infty)$.

In this case, $k_0 \in i\mathbb{R}$ and $\varkappa_0 = \sqrt{c^2 + k_0^2} \in i\mathbb{R}$ such that $\phi(x; k_0) = b_0 \psi(x; k_0) \to 0$ as $x \to +\infty$. The branch cut can be chosen for $\mathrm{Im}(k) \in [-c, c]$.

Soliton-RW interaction: trapped soliton

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$$u(x) = 2\mu_0^2 \mathrm{sech}^2(\mu_0(x - x_0)) + c^2 H(x),$$

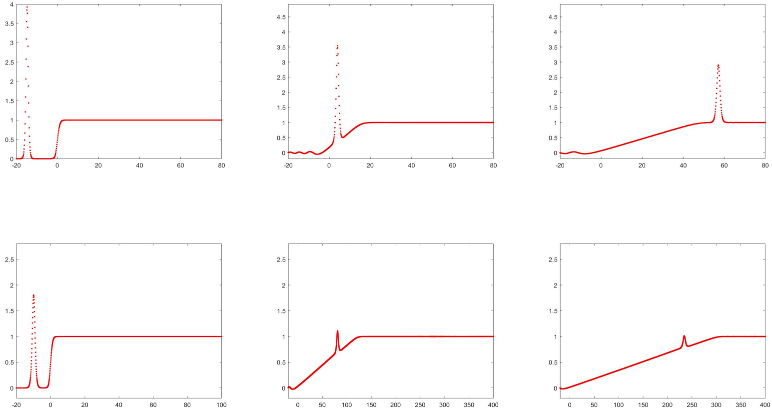
where $\mu_0 > 0$, $x_0 < 0$, and $H(x)$ is the Heaviside step function.

For every sufficiently large $x_0 \ll -1$, there exists a resonant pole $\lambda_0 \in \mathbb{C}$ with $\mathrm{Re}(\lambda_0) \approx -\mu_0^2$ if $\mu_0 \in (0, c)$.

In this case $k_0 \in \mathbb{C}$ and $\varkappa_0 = \sqrt{c^2 + k_0^2} \in \mathbb{C}$ satisfies $\mathrm{Re}(\varkappa_0) > 0$ and $\mathrm{Im}(\varkappa_0) < 0$ so that $\phi(x; k_0) = b_0 \psi(x; k_0) \to \infty$ as $x \to +\infty$. The branch cut can be chosen for $|\mathrm{Im}(k)| \in [c, \infty)$.

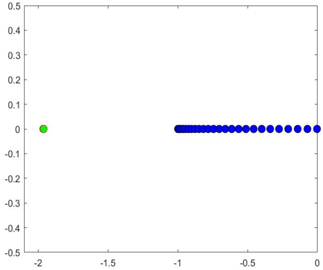
Numerical experiments

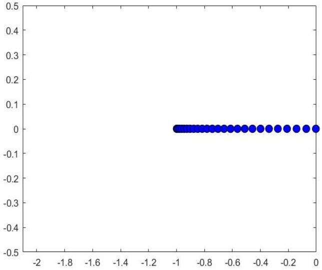
We use Zabusky–Kruskal scheme to recover transmission of a large soliton over the RW background and trapping of a small soliton



Numerical experiments

Lax spectrum contains an isolated eigenvalue for the transmitted soliton but does not contain any eigenvalues for the trapped soliton.





Numerical experiments

Let a^2 be the background (which depends on time t) and the solitary wave with parameter ν_0^2 is

$$u(t,x) = a^2 + 2\nu_0^2 \mathrm{sech}^2[\nu_0(x - 4\nu_0^2 t - 6a^2 t - x_0)].$$

Then, $\nu_0 = \sqrt{\mu_0^2 - a^2}$ by direct scattering, where μ_0^2 is parameter of the solitary waves at zero background. The soliton amplitude is

$$A = a^2 + 2\nu_0^2 = 2\mu_0^2 - a^2.$$

We can detect numerically the background parameter a^2 at time $t > 0$ from the location of the local maximum for the solitary wave.

Numerical experiments

In order to detect $a^2(t)$ for the RW background, we solve $u_t + 6uu_x = 0$ with

$$u(t,x) = \begin{cases} 0, & x < -\varepsilon, \\ (2\varepsilon + 6t)^{-1}(x+\varepsilon), & -\varepsilon \leq x \leq \varepsilon + 6t, \\ 1, & x > \varepsilon + 6t. \end{cases}$$

Let $\xi(t)$ be the numerically detected location of the solitary wave inside RW. Then,

$$a^2(t) = (2\varepsilon + 6t)^{-1}(\xi(t) + \varepsilon),$$

with which we compute the amplitude of the solitary waves

$$A(t) = 2\mu_0^2 - a^2(t).$$

Numerical experiments

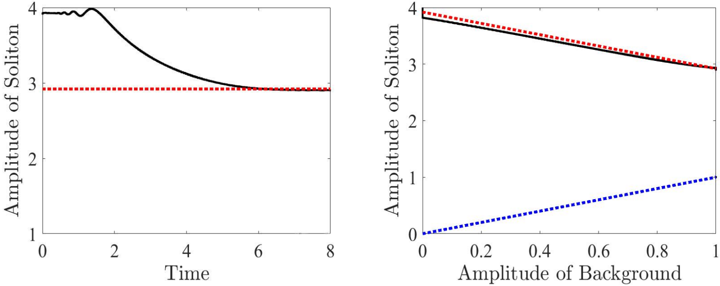


Figure: Data analysis for the transmitted soliton: (a) Amplitude of the solitary wave versus time (black) and the limiting amplitude $A_\infty = 2\mu_0^2 - c^2$ (red). (b) Amplitude of the solitary wave versus amplitude of the RW background detected numerically (black) and theoretically (red). The blue dots show the amplitude of the RW background.

Numerical experiments

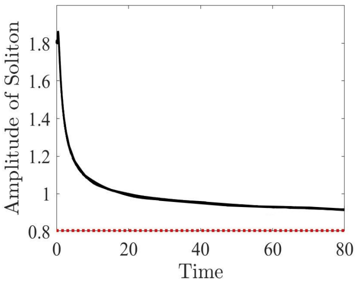


Figure: Data analysis for the trapped soliton: (a) Amplitude of the solitary wave versus time (black) and the limiting amplitude $A_\infty = 2\mu_0^2 - c^2$ (red). (b) Amplitude of the solitary wave versus amplitude of the RW background detected numerically (black) and theoretically (red). The blue dots show the amplitude of the RW background.

Summary on Soliton-RW interactions

- A transmitted soliton over the RW background can be generated by using the Darboux transformation which adds an isolated eigenvalue to the spectrum.
- No trapped soliton exists as it is related to resonant poles of the Schrödinger equation.
- The final amplitude of the transmitted soliton is determined by the initial amplitude. The amplitude of the trapped soliton slowly decays towards the amplitude of the RW background.

Soliton–DSW interactions

Since the DSW is modeled by the modulated traveling periodic wave, we consider the interaction of the soliton with a family of traveling periodic waves. The normalized traveling wave of the KdV equation:

$$u(t,x) = 2k^2 \mathrm{cn}^2(x - ct; k), \qquad c = 4(2k^2 - 1).$$

with the period $2K(k)$.

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This is the most general traveling periodic wave solution due to the scaling transformation

$$u(t,x) \quad \Rightarrow \quad \alpha^2 u(\alpha^3 t, \alpha x), \quad \alpha > 0,$$

the Galilean transformation

$$u(t,x) \quad \Rightarrow \quad \beta + u(t, x - 6\beta t), \quad \beta \in \mathbb{R},$$

and the translational symmetry: $u(t,x) \to u(t, x - \gamma)$, $\gamma \in \mathbb{R}$.

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sn, cn, and dn are real-valued Jacobi elliptic functions with

$$\mathrm{sn}^2 + \mathrm{cn}^2 = 1, \quad \mathrm{dn}^2 + k^2 \mathrm{sn}^2 = 1.$$

Parameter $k \in (0, 1)$ is the elliptic modulus. Elliptic functions reduce to the trigonometric functions as $k \to 0$ and to the hyperbolic functions as $k \to 1$.

Soliton–DSW interactions

Question: What are properties of the solitary wave propagating on the traveling periodic wave background?

E. Kuznetsov, A. Mikhailov, JETP **40** (1974) 855
F. Gesztesy, R. Svirsky, Memoirs AMS **118** (1995) 1–88
X.R. Hu, S.Y. Lou, Y. Chen, Phys. Rev. E **85** (2012) 056607
A. Nakayashiki, Lett. Math. Phys. **111** (2021) 85

Due to periodicity of the interactions between the solitary wave and the periodic background, we refer to these solutions as *breathers* and study their properties (speed, localization, phase shift).

M. Hoefer, A. Mucalica and D.E. Pelinovsky, KdV breathers on cnoidal wave background, J. Physics A: Mathem. Theor. 56 (2023) 185701

Soliton–DSW interactions

One can again use the Darboux–Backlund transformation

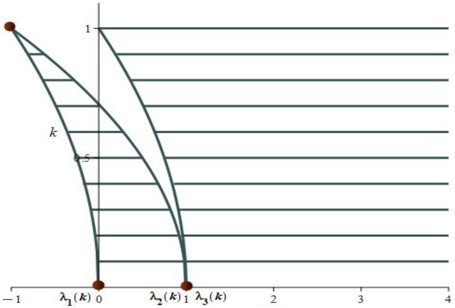
$$\hat{u} := u + 2\frac{\partial^2}{\partial x^2}\log(v_0),$$

where $v_0(t,x) = v(x-ct)e^{\omega t}$ is a solution of the Lamé equation

$$v''(x) + 2k^2\mathrm{cn}^2(x;k)v(x) + \lambda v(x) = 0$$

with some uniquely determined $\omega = \omega(\lambda)$.

Soliton–DSW interactions

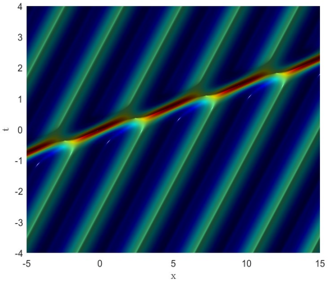


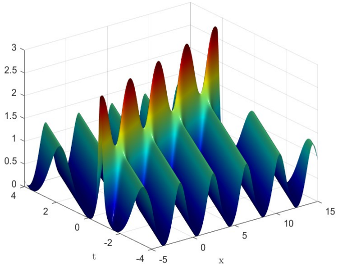
Bright breathers correspond to λ in semi-infinite gap.

Dark breathers correspond to λ in the finite gap.

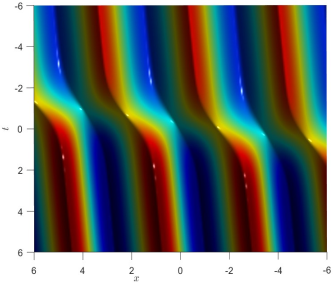
Soliton–DSW interactions

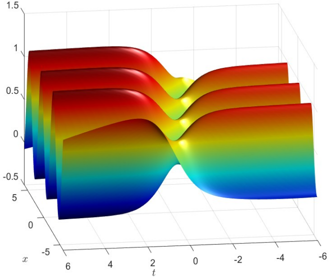
Bright breather





Dark breather





Construction of breathers

The Lamé equation

$$v''(x) + 2k^2 \mathrm{cn}^2(x;k) v(x) + \lambda v(x) = 0$$

is solved with the explicit functions

$$v_\pm(x) = \frac{H(x \pm \alpha)}{\Theta(x)} e^{\mp x Z(\alpha)},$$

where $\lambda = 1 - 2k^2 + k^2 \mathrm{cn}(\alpha;k)$ and

$$H(x) = \theta_1\left(\frac{\pi x}{2K}\right) = 2\sum_{n=1}^{\infty} (-1)^{n-1} q^{(n-1/2)^2} \sin(2n-1)\left(\frac{\pi x}{2K}\right),$$

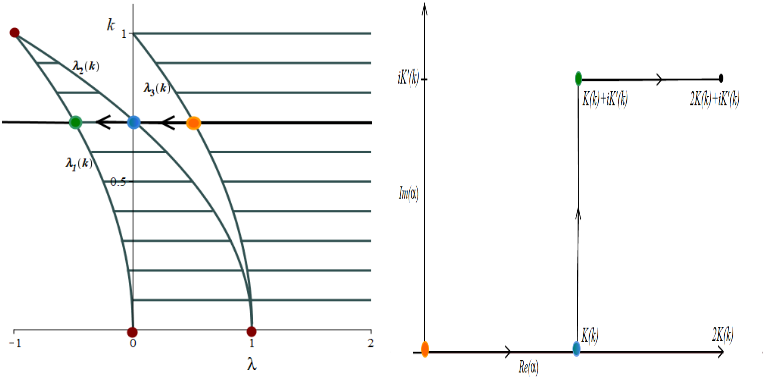
$$\Theta(x) = \theta_4\left(\frac{\pi x}{2K}\right) = 1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2} \cos(2n)\left(\frac{\pi x}{2K}\right).$$

D. F. Lawden, *Elliptic Functions and Applications*, (Springer, 2013)

Construction of breathers

Nontrivial relation between λ and α:

$$\lambda = 1 - 2k^2 + k^2 \mathrm{cn}(\alpha; k)$$



Construction of breathers

The time evolution of the eigenfunctions follows from the separation of variables in the Lax system:

$$v_\pm(t,x) = \frac{H(x-ct\pm\alpha)}{\Theta(x-ct)} e^{\mp(x-ct)Z(\alpha)\mp t\omega(\alpha)},$$

where $\omega(\alpha)$ is found from the time evolution equation:

$$\omega(\alpha) = (c_0 + 4\lambda - 2\phi_0(x))\left[Z(\alpha) \pm Z(x) \mp \frac{H'(x\pm\alpha)}{H(x\pm\alpha)}\right] \mp \phi_0'(x),$$

or equivalently at $x = 0$:

$$\omega(\alpha) = 4(\lambda + k^2 - 1)\left[\frac{\Theta'(\alpha)}{\Theta(\alpha)} - \frac{H'(\alpha)}{H(\alpha)}\right].$$

Construction of breathers

Darboux transformation for λ in the semi-infinite gap is applied with

$$v_0(x,t) = c_+ v_+(x,t) + c_- v_-(x,t)$$

where $v_\pm(x,t) > 0$ and $c_\pm > 0$. We can use that

$$k^2 \mathrm{cn}^2(x,k) = k^2 - 1 + \frac{E(k)}{K(k)} + \partial_x^2 \log \Theta(x)$$

and obtain the new solution

$$\hat{u} = u + 2\frac{\partial^2}{\partial x^2}\log(v_0) = 2\left[k^2 - 1 + \frac{E(k)}{K(k)}\right] + 2\partial_x^2 \log \tau,$$

$$\tau = \Theta(x - c_0 t + \alpha_b) e^{\kappa_b (x - c_b t + x_0)} + \Theta(x - c_0 t - \alpha_b) e^{-\kappa_b (x - c_b t + x_0)}$$

with uniquely defined parameters $c_b > c_0$, $\kappa_b > 0$, and $\alpha_b \in [0, K]$ for each λ.

Construction of breathers

Darboux transformation for λ in the finite gap is applied with

$$v_0(x,t) = c_+ v_+(x,t) + c_- v_-(x,t)$$

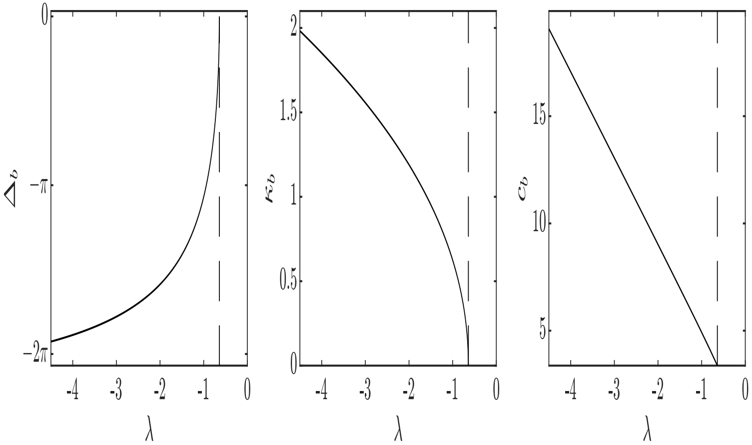
but $v_\pm(x,t)$ are sign-indefinite. However, translation of the new solution $\hat{u} = \hat{u}(x + iK', t)$ yields a bounded solution

$$\hat{u} = u + 2\frac{\partial^2}{\partial x^2}\log(v_0) = 2\left[k^2 - 1 + \frac{E(k)}{K(k)}\right] + 2\partial_x^2 \log\tau,$$

$$\tau = \Theta(x - c_0 t + \alpha_d)e^{-\kappa_d(x - c_d t + x_0)} + \Theta(x - c_0 t - \alpha_d)e^{\kappa_d(x - c_d t + x_0)}$$

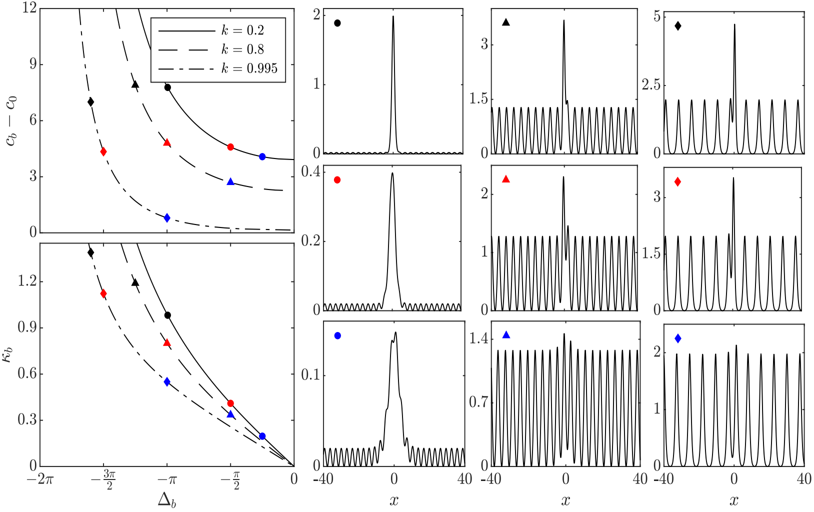
with uniquely defined parameters $c_d < c_0$, $\kappa_d > 0$, and $\alpha_d \in [0, K]$ for each λ.

Bright breathers

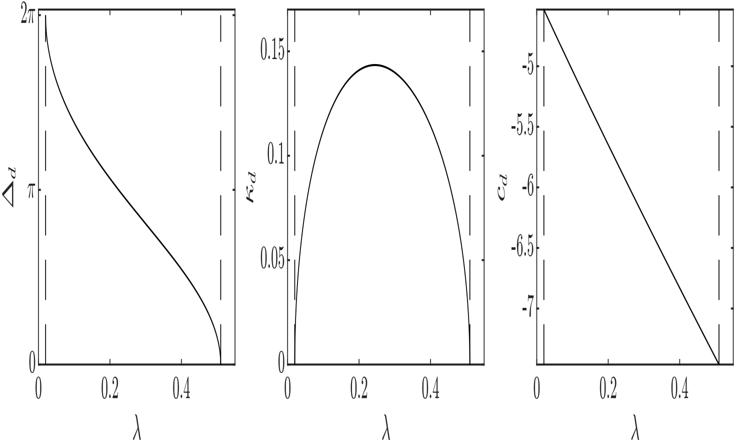


Here $\Delta_b = 2\pi\alpha_b/K(k)$ is normalized phase shift.
We can prove $\Delta_b'(\lambda) > 0$, $\kappa_b'(\lambda) < 0$, and $c_b > c_0$.

Bright breathers

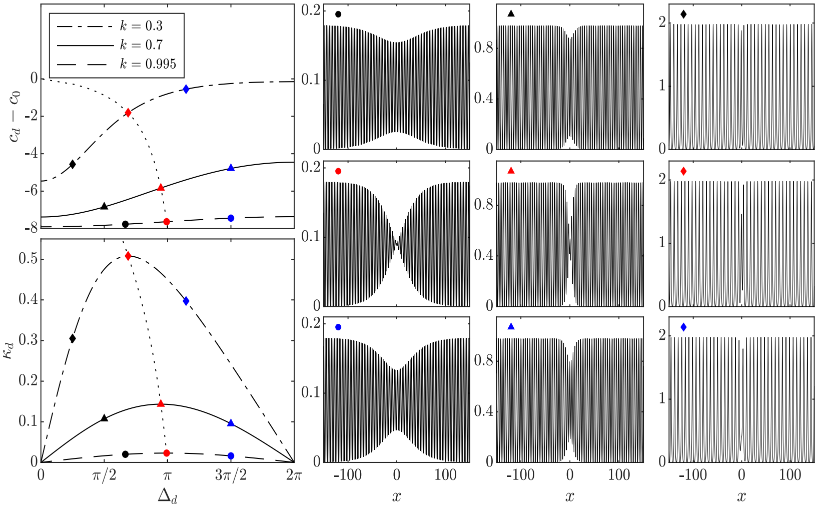


Dark breathers



Here $\Delta_d = 2\pi\alpha_d/K(k)$ is normalized phase shift.
We can prove $\Delta_d'(\lambda) < 0$, $\max \kappa_d(\lambda)$, and $c_d < c_0$.

Dark breathers



Dark breathers

Limits $k \to 0$ and $k \to 1$ can be studied very precisely thanks to availability of the exact solutions in the closed form.

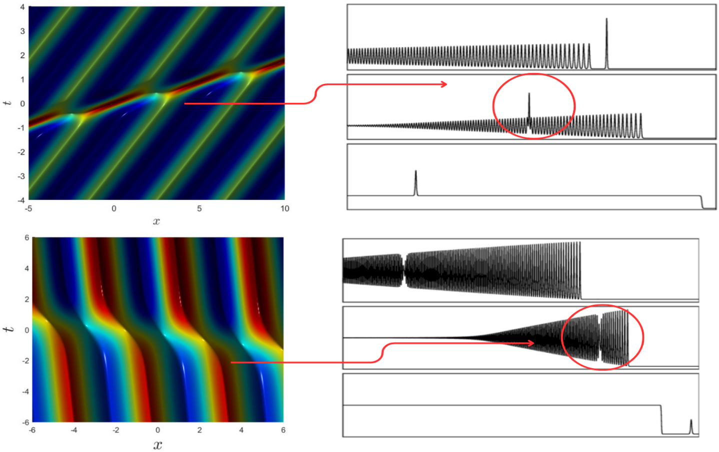
The limit $k \to 1$ gives the 2-soliton solutions with two solitons of two different speeds.

The limit $k \to 0$ for the bright breather of KdV yields the 1-soliton solution.

The limit $k \to 0$ for the dark breathers of KdV yields the dark solitons of NLS in agreement with

- Zakharov, V. E.; Kuznetsov, E. A. Multiscale expansions in the theory of systems integrable by the inverse scattering transform. Phys. D 18 (1986), no. 1-3, 455–463.

Comparison between breathers and Soliton-DSW



Summary on Soliton-DSW interactions

- A transmitted soliton over the DSW background can be related to the bright breather solution.
- A trapped soliton in the DSW background can be related to the dark breather solution.
- Parameters of the breather solutions are computed explicitly in terms of the Jacobi elliptic functions.

Transmission of solitons in other integrable models

Defocusing NLS and modified KDV equations are also physically interesting models where similar problems are relevant and similar solutions can be constructed. For the defocusing MKDV,

$$u_t - 6u^2 u_x + u_{xxx} = 0,$$

the normalized traveling wave is

$$u(x,t) = \phi_0(x + c_0 t), \quad \phi_0(x) = k\,\mathrm{sn}(x;k), \quad c_0 = 1 + k^2,$$

although it is not the most general solution of the MKDV equation. It corresponds to

$$\phi''' - 6\phi^2\phi' + c\phi' = 0 \quad \Rightarrow \quad \phi'' - 2\phi^3 + c\phi = b,$$

with $b = 0$. The most general solution has $b \neq 0$.

Transmission of solitons in other integrable models

The spectral problem is self-adjoint

$$\varphi_x = \begin{pmatrix} i\zeta & u \\ u & -i\zeta \end{pmatrix} \varphi,$$

hence the spectrum associated with the traveling periodic wave is purely real: $\zeta \in \mathbb{R}$.

Transmission of solitons in other integrable models

Explicit solutions for eigenfunctions are available:

$$p(x;z) = e^{s(z)x} e^{-\frac{i\pi x}{4K}} \frac{\theta_1\left(\frac{\pi(x-iz)}{2K}\right)}{\theta_4\left(\frac{\pi x}{2K}\right)\theta_4\left(\frac{i\pi z}{2K}\right)}$$

$$q(x;z) = e^{s(z)x} e^{-\frac{i\pi x}{4K}} \frac{\theta_4\left(\frac{\pi(x-iz)}{2K}\right)}{\theta_4\left(\frac{\pi x}{2K}\right)\theta_1\left(\frac{i\pi z}{2K}\right)},$$

where

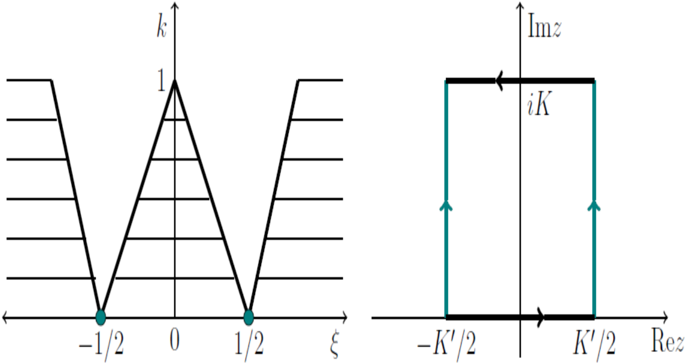
$$s(z) = \frac{1}{2}Z(iz) - \frac{1}{2}Z(iz'), \quad z' = K'(k) - z$$

and

$$\zeta(z) = \frac{1}{2}\mathrm{dn}(iz)\mathrm{dn}(iz').$$

D. A. Takahashi, Phys. Rev E. **93** (2016) 062224

Transmission of solitons in other integrable models



Transmission of solitons in other integrable models

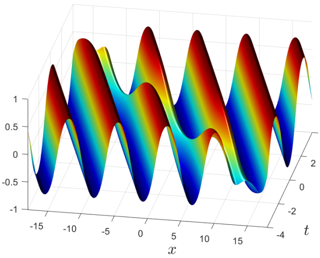
New solution for soliton on the periodic background is obtained with the Darboux transformation:

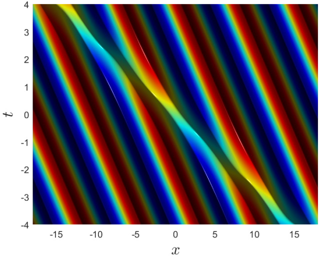
$$\hat{u} = u - \frac{4i\zeta_0 p_0 q_0}{p_0^2 - q_0^2},$$

where $\varphi = (p_0, q_0)$ is a superposition of two linearly independent solutions with $q_0 = \bar{p}_0$. However, the new solution is singular and a bounded solution is obtained after the transformation: $x \to x + iK'$.

Transmission of solitons in other integrable models

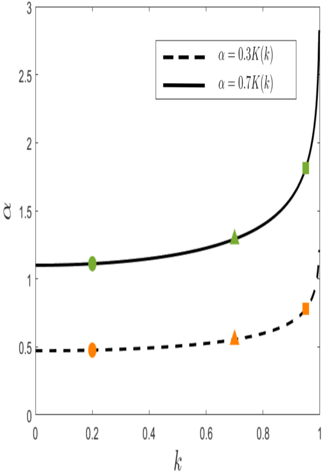
The solution surface of the breather on the periodic background:

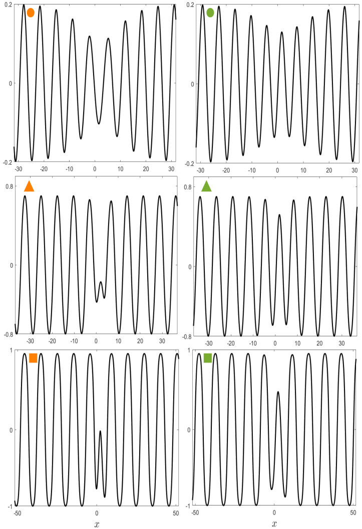




Transmission of solitons in other integrable models

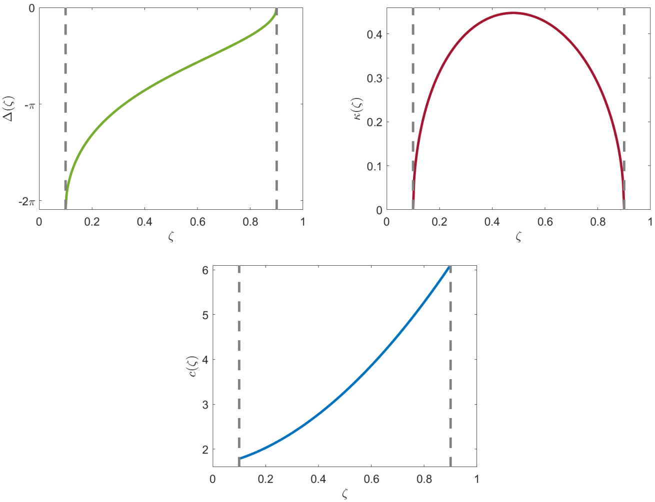
The family of breathers on the periodic background:





Transmission of solitons in other integrable models

Characteristics of the dark breather (phase shift, localization, speed)



Transmission of solitons in other integrable models

Other models with modulationally stable periodic waves?

Benjamin–Ono equation, intermediate Long–Wave equation, Kadomtsev–Petviashvili-II equation?

Many thanks for your attention!