Solitons on the rarefactive wave background

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after inspirations from Mark Hoefer (University of Colorado at Boulder)

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Introduction

We are dealing with the canonical model for the shallow water waves, the Korteweg–de Vries (KdV) equation:

$$u_t + 6uu_x + u_{xxx} = 0, \qquad \text{(KdV)}$$

subject to the boundary conditions

$$\lim_{x\to-\infty} u(t,x) = 0, \qquad \lim_{x\to+\infty} u(t,x) = c^2. \qquad \text{(BC)}$$

Applications: tidal bores, earthquake-generated waves

G. A. El, Adv. Fluid Mech. **47** (2007) 19-53

G. A. El and M. A. Hoefer, Physica D **333** (2016) 11-65

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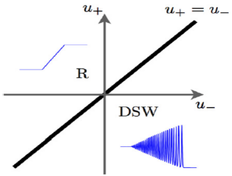
The step-like initial data results in the appearance of a rarefactive wave (RW) for $t > 0$ and a dispersive shock wave (DSW) for $t < 0$.



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For simplicity, we consider only the RW case ($t > 0$) but would like to consider interaction of solitary waves with the RW.

M. D. Maiden, D. V. Anderson, A. A. Franco, G. A. El, and M. A. Hoefer, Phys. Rev. Lett. **120** (2018) 144101
P. Sprenger, M. A. Hoefer, and G. A. El, Phys. Rev. E **97** (2018) 032218
T. Congy, G. A. El and M. A. Hoefer, J. Fluid Mech. **875** (2019) 1145–1174
K. van der Sande, G. A. El and M. A. Hoefer, J. Fluid Mech. **928** (2021) A21

State of the art

Depending on the initial amplitude of a solitary wave, it is either transmitted over or trapped inside the RW background.

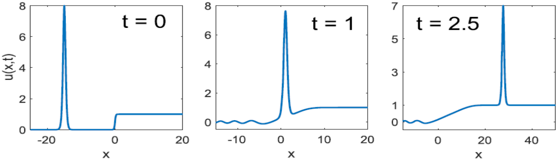

FIG. 1. Tunneling soliton example for $\kappa_1 = 2$, $c = 1$, $x_0 = -15$.

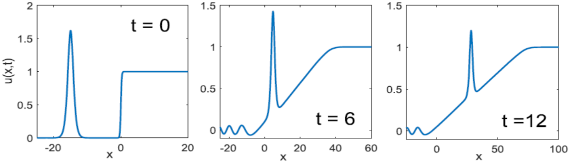

FIG. 2. Trapped soliton example for $\kappa_0 = 0.9$, $c = 1$, $x_0 = -15$.

M. J. Ablowitz, X. D. Luo, and J. T. Cole, J. Math. Phys. **59** (2018), 091406

State of the art

This phenomenon was interpreted from the inverse scattering method:

$$\mathcal{L}v = \lambda v, \qquad \mathcal{L} := -\frac{\partial^2}{\partial x^2} - u$$

and

$$\frac{\partial v}{\partial t} = \mathcal{M}v, \qquad \mathcal{M} := -3u_x - 6u\frac{\partial}{\partial x} - 4\frac{\partial^3}{\partial x^3},$$

where $\lim_{x\to-\infty} u(t,x) = 0$ and $\lim_{x\to+\infty} u(t,x) = c^2$.

- Transmitted soliton corresponds to an isolated eigenvalue of $\mathcal{L}$.
- Trapped soliton corresponds to a “pseudo–embedded" eigenvalue inside the continuous spectrum of $\mathcal{L}$ in $[-c^2, 0]$

M. J. Ablowitz, X. D. Luo, and J. T. Cole, J. Math. Phys. **59** (2018), 091406

State of the art

The rigorous IST method was applied for the step-like BC in the DSW case ($t < 0$):

- N solitons added as poles in the IST method scatter towards zero boundary conditions as t evolves.
- Phase shifts of the N solitons were appropriately computed.
- These N solitons are considered to be transmitted solitons over the DSW background.
- No trace of trapped solitons appear in the IST method.

I. Egorova, Z. Gladka, V. Kotlyarov, and G. Teschl,
Nonlinearity **26** (2013) 1839–1864

I. Egorova, J. Michor, and G. Teschl, arXiv: 2109.08423 (2021)

Transmitted soliton via Darboux transformation

Let u be a solution of the KdV equation and v_0 be a real solution of the Lax equations for $\lambda = \lambda_0 \in \mathbb{R}$ such that $v_0 \neq 0$. Then,

$$\hat{u} := u + 2\frac{\partial^2}{\partial x^2}\log(v_0)$$

is a new solution of the KdV equation.

If λ_0 is below the bottom of the spectrum of $\mathcal{L} = -\partial_x^2 - u$, then $v_0 \neq 0$ everywhere by Sturm's nodal theory.

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Let $u(x) = 0$ for $x < 0$ and $u(x) = c^2$ for $x > 0$ at initial time $t = 0$. Pick $\lambda_0 = -\mu_0^2 < -c^2$ and obtain

$$v_0(x) = \begin{cases} e^{\mu_0(x-x_0)} + e^{-\mu_0(x-x_0)}, & x < 0, \\ c_1 e^{\nu_0 x} + c_2 e^{-\nu_0 x}, & x > 0, \end{cases}$$

where $\nu_0 := \sqrt{\mu_0^2 - c^2} > 0$, x_0 is arbitrary, and (c_1, c_2) are uniquely found from the continuity of v_0 and v_0' across $x = 0$.

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$$\hat{u} := u + 2\frac{\partial^2}{\partial x^2}\log(v_0)$$

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The new solution is

$$\hat{u}(x) = 2\mu_0^2 \operatorname{sech}^2[\mu_0(x - x_0)], \quad x < 0$$

and

$$\hat{u}(x) = c^2 + 4\nu_0^2 \frac{\nu_0^2 + \mu_0^2 + (\nu_0^2 - \mu_0^2)\cosh(2\mu_0 x_0)}{[(\nu_0 + \mu_0)\cosh(\nu_0 x - \mu_0 x_0) + (\nu_0 - \mu_0)\cosh(\nu_0 x + \mu_0 x_0)]^2},$$

for $x < 0$ and $x > 0$ respectively. The solution is bounded if $x_0 \leq 0$.

Transmitted soliton via Darboux transformation

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$$\hat{u} := u + 2\frac{\partial^2}{\partial x^2}\log(v_0)$$

is a new solution of the KdV equation.

- The solitary wave on the step background decays differently as $x \to -\infty$ (decay rate is μ_0) and as $x \to +\infty$ (decay rate is $\nu_0 = \sqrt{\mu_0^2 - c^2}$).
- The Lax spectrum of the new solution is $[-c^2, \infty)$ and a simple isolated eigenvalue $\lambda_0 = -\mu_0^2 < -c^2$.

Trapped soliton via direct scattering

Some evidences that no trapped solitons actually exist.

- Eigenfunctions of $\mathcal{L}v = \lambda_0 v$ are bounded but not decaying if $\lambda \in [-c_0^2, \infty)$. No embedded eigenvalues exist if

$$u(x) \to c^2 \quad \text{as} \;\; x \to +\infty \qquad \text{rapidly}$$

- Darboux transformation does not produce any bounded solutions if $\lambda_0 \in [-c_0^2, \infty)$.

Trapped soliton via direct scattering

Consider a linear superposition of a soliton and the step function:

$$u(x) = 2\mu_0^2 \mathrm{sech}^2(\mu_0(x - x_0)) + c^2 H(x),$$

where $\mu_0 > 0$ and $x_0 < 0$.

Eigenfunctions of $\mathcal{L}v = \lambda v$ with $\lambda = k^2$ are known explicitly:

$$\phi(x;k) = e^{-ikx}\left[1 - \frac{i\mu_0}{k + i\mu_0} e^{\mu_0(x-x_0)} \mathrm{sech}(\mu_0(x - x_0))\right], \quad x < 0$$

and

$$\psi(x;k) = e^{i\varkappa x}\left[1 - \frac{i\mu_0}{\varkappa + i\mu_0} e^{-\mu_0(x-x_0)} \mathrm{sech}(\mu_0(x - x_0))\right], \quad x > 0,$$

where $\varkappa := \sqrt{c^2 + k^2}$.

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where $\mu_0 > 0$ and $x_0 < 0$.

The scattering data are obtained from the scattering relation:

$$\phi(x;k) = a(k)\overline{\psi}(x;k) + b(k)\psi(x;k), \quad x \in \mathbb{R},$$

where $\bar{\psi}$ is obtained from ψ by reflection $\varkappa \mapsto -\varkappa$.

Straightforward computation yields:

$$a(k) = \frac{(\varkappa + k)\left(\varkappa k + \mu_0^2 + i\mu_0(\varkappa - k)\tanh(\mu_0 x_0)\right)}{2\varkappa(\varkappa + i\mu_0)(k + i\mu_0)}.$$

Trapped soliton via direct scattering

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where $\mu_0 > 0$ and $x_0 < 0$.

$\phi(x;k)$ and $a(k)$ can be continued analytically for $k \in \mathbb{C}$, $\mathrm{Im}(k) > 0$. However, $k = ic$ is a branch point for $\kappa := \sqrt{c^2 + k^2}$. Branch cuts must be defined on $i\mathbb{R}$ either for $\mathrm{Im}(k) \in [-c, c]$ or for $|\mathrm{Im}(k)| \in [c^2, \infty)$.

- If $a(k_0) = 0$ with $\mathrm{Im}(k_0) \in (c, \infty)$, then $\varkappa_0 := \sqrt{c^2 + k_0^2}$ satisfies $\mathrm{Im}(\varkappa_0) > 0$ and $\phi(x;k_0) = b_0 \psi(x;k_0) \to 0$ as $x \to +\infty$. This yields **the isolated eigenvalue** $\lambda_0 := k_0^2$, for which the branch cut can be chosen for $\mathrm{Im}(k) \in [-c, c]$.

Trapped soliton via direct scattering

Consider a linear superposition of a soliton and the step function:

$$u(x) = 2\mu_0^2 \text{sech}^2(\mu_0(x - x_0)) + c^2 H(x),$$

where $\mu_0 > 0$ and $x_0 < 0$.

$\phi(x;k)$ and $a(k)$ can be continued analytically for $k \in \mathbb{C}$, $\text{Im}(k) > 0$. However, $k = ic$ is a branch point for $\kappa := \sqrt{c^2 + k^2}$. Branch cuts must be defined on $i\mathbb{R}$ either for $\text{Im}(k) \in [-c, c]$ or for $|\text{Im}(k)| \in [c^2, \infty)$.

- If $a(k_0) = 0$ with $\text{Im}(k_0) \in (0, c)$, then $\varkappa_0 := \sqrt{c^2 + k_0^2}$ satisfies $\text{Re}(\varkappa_0) > 0$. If in addition, $\text{Im}(\varkappa_0) < 0$, then $\phi(x;k_0) = b_0 \psi(x;k_0) \to \infty$ as $x \to +\infty$. This yields **the resonant pole** $\lambda_0 := k_0^2$, for which the branch cut is for $|\text{Im}(k)| \in [c, \infty)$.

Trapped soliton via direct scattering

Consider a linear superposition of a soliton and the step function:

$$u(x) = 2\mu_0^2 \mathrm{sech}^2(\mu_0(x - x_0)) + c^2 H(x),$$

where $\mu_0 > 0$ and $x_0 < 0$.

By using

$$a(k) = \frac{(\varkappa + k)\left(\varkappa k + \mu_0^2 + i\mu_0(\varkappa - k)\tanh(\mu_0 x_0)\right)}{2\varkappa(\varkappa + i\mu_0)(k + i\mu_0)},$$

we are looking for roots $k \in \mathbb{C}$ with $\mathrm{Im}(k) > 0$ of equation

$$\varkappa k + \mu_0^2 + i\mu_0(\varkappa - k)\tanh(\mu_0 x_0) = 0.$$

If $x_0 \to -\infty$, $k = i\mu_0$ is a simple root of this equation.

Trapped soliton via direct scattering

Consider a linear superposition of a soliton and the step function:

$$u(x) = 2\mu_0^2 \text{sech}^2(\mu_0(x - x_0)) + c^2 H(x),$$

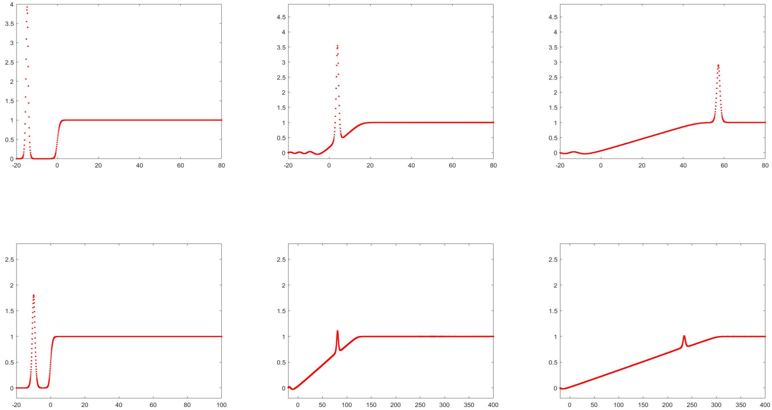
where $\mu_0 > 0$ and $x_0 < 0$.

Assume $x_0 \ll -1$.

- ▷ An isolated eigenvalue $\lambda_0 \in (-\infty, -c^2)$ persists near $-\mu_0^2$ if $\mu_0 \in (c, \infty)$.
- ▷ An embedded eigenvalue $\lambda_0 \in (-c^2, 0)$ moves to a resonant pole $\lambda = ik_0$ with $\text{Re}(k_0) < 0$, $\text{Im}(k_0) > 0$, and $\text{Im}(\varkappa_0) < 0$ if $\mu_0 \in (0, c)$. Resonant poles do not correspond eigenvalues.

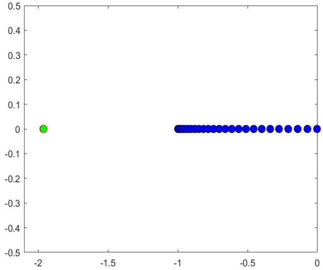
Numerical experiments

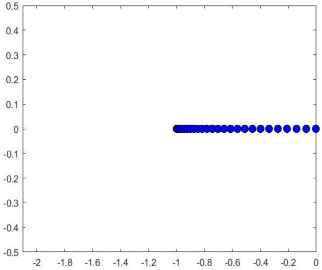
We use Zabusky–Kruskal scheme to recover transmission of a large soliton over the RW background and trapping of a small soliton



Numerical experiments

Lax spectrum contains an isolated eigenvalue for the transmitted soliton but does not contain any eigenvalues for the trapped soliton.





Numerical experiments

Let a^2 be the background (which depends on time t) and the solitary wave with parameter ν_0^2 is

$$u(t,x) = a^2 + 2\nu_0^2 \text{sech}^2[\nu_0(x - 4\nu_0^2 t - 6a^2 t - x_0)].$$

Then, $\nu_0 = \sqrt{\mu_0^2 - a^2}$ by direct scattering, where μ_0^2 is parameter of the solitary waves at zero background. The soliton amplitude is

$$A = a^2 + 2\nu_0^2 = 2\mu_0^2 - a^2.$$

Numerical experiments

In order to detect $a^2(t)$ for the RW background, we solve $u_t + 6uu_x = 0$ with

$$u(t,x) = \begin{cases} 0, & x < -\varepsilon, \\ (2\varepsilon + 6t)^{-1}(x+\varepsilon), & -\varepsilon \leq x \leq \varepsilon + 6t, \\ 1, & x > \varepsilon + 6t. \end{cases}$$

If $\xi(t)$ is the numerically detected location of the solitary wave inside RW, then

$$a^2(t) = (2\varepsilon + 6t)^{-1}(\xi(t) + \varepsilon),$$

with which we compute the amplitude of the solitary waves

$$A(t) = 2\mu_0^2 - a^2(t).$$

Numerical experiments

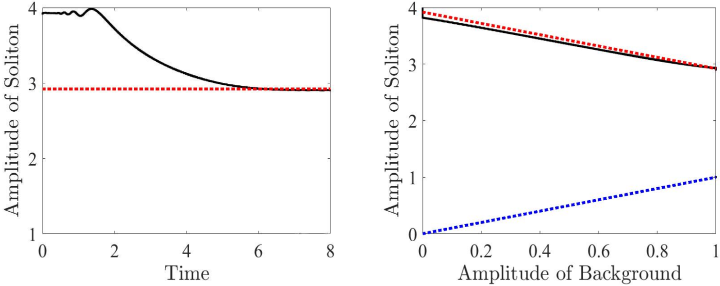


Figure: Data analysis for the transmitted soliton: (a) Amplitude of the solitary wave versus time (black) and the limiting amplitude $A_\infty = 2\mu_0^2 - c^2$ (red). (b) Amplitude of the solitary wave versus amplitude of the RW background detected numerically (black) and theoretically (red). The blue dots show the amplitude of the RW background.

Numerical experiments

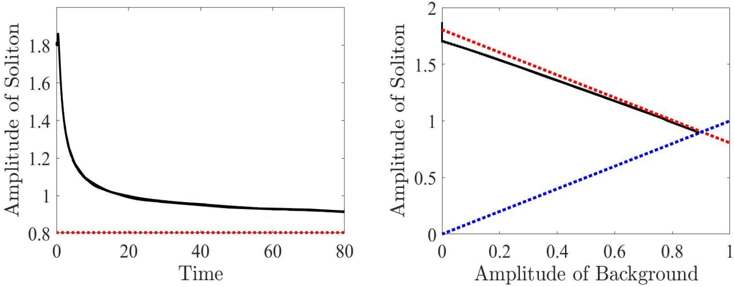


Figure: Data analysis for the trapped soliton: (a) Amplitude of the solitary wave versus time (black) and the limiting amplitude $A_\infty = 2\mu_0^2 - c^2$ (red). (b) Amplitude of the solitary wave versus amplitude of the RW background detected numerically (black) and theoretically (red). The blue dots show the amplitude of the RW background.

Summary

- Transmitted solitary wave over the RW background can be generated by using the Darboux transformation which adds an isolated eigenvalue.
- No trapped solitary wave exists as it is related to resonant poles of the Schrödinger equation.
- The amplitude of the transmitted soliton is determined by the initial amplitude. The amplitude of the trapped soliton decays to the amplitude of the RW background.
- Open question: What is a new solution generated by the Darboux transformation from a complex-conjugate pair of resonant poles?

Ongoing project with Mark Hoefer

The KdV equation

$$u_t + 6uu_x + u_{xxx} = 0, \qquad \text{(KdV)}$$

has a family of traveling periodic wave solutions

$$u(t,x) = 2k^2 \mathrm{cn}^2(x - ct; k), \qquad c = 4(2k^2 - 1).$$

Question: What are the solitary waves propagating on the traveling periodic wave background?

E. Kuznetsov, A. Mikhailov, JETP **40** (1974) 855
F. Gesztesy, R. Svirsky, Memoirs AMS **118** (1995) 1–88
X.R. Hu, S.Y. Lou, Y. Chen, Phys. Rev. E **85** (2012) 056607
A. Nakayashiki, Lett. Math. Phys. **111** (2021) 85

Ongoing project with Mark Hoefer

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$$u(t,x) = 2k^2 \mathrm{cn}^2(x - ct; k), \qquad c = 4(2k^2 - 1).$$

One can again use the Darboux–Backlund transformation

$$\hat{u} := u + 2\frac{\partial^2}{\partial x^2}\log(v_0),$$

where $v_0(t,x) = v(x-ct)e^{\omega t}$ is a solution of the Lamé equation

$$v''(x) + 2k^2\mathrm{cn}^2(x;k)v(x) + \lambda v(x) = 0$$

with some uniquely determined $\omega = \omega(\lambda)$.

Ongoing project with Mark Hoefer

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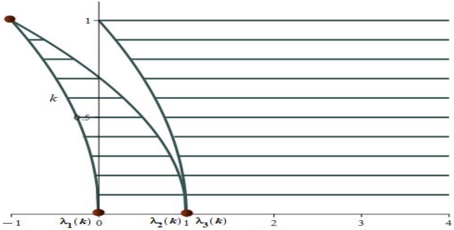

Bright solitons correspond to λ in semi-infinite gap.

Dark solitons correspond to λ in the finite gap.

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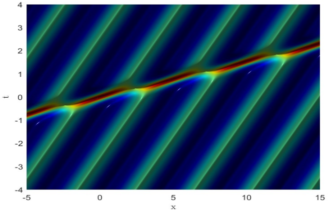
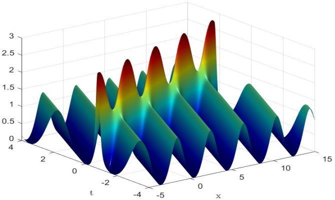
Bright soliton propagation





The KdV equation

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$$u(t,x) = 2k^2 \mathrm{cn}^2(x - ct; k), \qquad c = 4(2k^2 - 1).$$

Dark soliton propagation