Nonlinear instabilities of solitons and breathers

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Stability of stationary states in Hamiltonian systems

Consider an abstract Hamiltonian dynamical system

$$\frac{du}{dt} = J \, \mathrm{grad} H(u), \quad u(t) \in X$$

where $X \subset L^2$ is a phase space, $J^+ = -J$ is a bounded invertible operator for the symplectic structure, and $H : X \to \mathbb{R}$ is the Hamilton function.

- Assume existence of the stationary state $u_0 \in X$ such that $\mathrm{grad} H(u_0) = 0$.
- Perform linearization $u(t) = u_0 + v e^{\lambda t}$, where λ is the spectral parameter and $v \in X$ satisfies the spectral problem

$$J H''(u_0) v = \lambda v,$$

where $H''(u_0) : X \to L^2$ is a self-adjoint Hessian operator.

Spectral stability

Consider the spectral problem:

$$JH''(u_0)v = \lambda v, \quad v \in X.$$

Assumptions:

- The spectrum of $H''(u_0)$ is positive except for finitely many negative and zero eigenvalues of finite multiplicity.
- The continuous wave spectrum of $JH''(u_0)$ is purely imaginary.
- Multiplicity of the zero eigenvalue of $JH''(u_0)$ is given by the number of parameters in u_0 (symmetries).

Question: Is there a relation between unstable eigenvalues of $JH''(u_0)$ and negative eigenvalues of $H''(u_0)$?

Example: NLS equation

Consider the nonlinear Schrödinger equation

$$iu_t = -u_{xx} + V(x)u + |u|^2 u, \quad x \in \mathbb{R},$$

V is an external potential and $u(t,x) = u_0(x)e^{-i\omega t}$ is a stationary state.

- u_0 is a critical point of the conserved energy:

$$H(u) = \int_{\mathbb{R}} \left(|u_x|^2 + V|u|^2 - \omega|u|^2 + \frac{1}{2}|u|^4 \right) dx.$$

- The self-adjoint Hessian operator $H''(u_0)$ is given by

$$H''(u_0) = \left[\begin{array}{cc} -\partial_x^2 + V - \omega + 2|u_0|^2 & u_0^2 \\ \bar{u}_0^2 & -\partial_x^2 + V - \omega + 2|u_0|^2 \end{array} \right].$$

- The bounded invertible operator J is

$$J = \left[\begin{array}{cc} i & 0 \\ 0 & -i \end{array} \right].$$

Example: nonlinear Dirac equation

Consider the nonlinear Dirac equation

$$\begin{cases} i(u_t + u_x) + v + (|u|^2 + 2|v|^2)u = 0, \\ i(v_t - v_x) + u + (2|u|^2 + |v|^2)v = 0, \end{cases}$$

where $(u,v)(t,x) = (u_0, v_0)(x)e^{-i\omega t}$ is a stationary state.

- (u_0, v_0) is a critical point of the conserved energy:

$$H(u,v) = \int_{\mathbb{R}} (u_x\bar{u} - u\bar{u}_x - v_x\bar{v} + v\bar{v}_x + v\bar{u} + u\bar{v})\, dx + \int_{\mathbb{R}} \left(\omega(|u|^2 + |v|^2) - \frac{1}{2}|u|^4 - 2|u|^2|v|^2 - \frac{1}{2}|v|^4\right) dx.$$

- The continuous wave spectrum of $H''(u_0, v_0)$ is sign-indefinite and hence violates assumptions of the theory.

Main question

Question: Is there a relation between unstable eigenvalues of $JH''(u_0)$ and negative eigenvalues of $H''(u_0)$?

For a gradient system:

$$\frac{du}{dt} = -\mathrm{grad} F(u) \quad \Rightarrow \quad \lambda v = -F''(u_0) v,$$

Theorem

The number of unstable eigenvalues of $-F''(u_0)$ equals the number of negative eigenvalues of $F''(u_0)$.

The relation is less straightforward in a Hamiltonian system

$$\lambda v = JH''(u_0) v, \quad v \in X.$$

Symmetry: If λ is an eigenvalue, so is $-\lambda$, $\bar{\lambda}$, and $-\bar{\lambda}$.

Example: two coupled oscillators

Question: Is there a relation between unstable eigenvalues of $JH''(u_0)$ and negative eigenvalues of $H''(u_0)$?

Consider energy

$$H = \frac{1}{2}(y_1^2 + y_2^2) + \frac{1}{2}(\omega_1^2 x_1^2 + \omega_2^2 x_2^2)$$

The quadratic form for H has four positive eigenvalues.

The two oscillators are stable:

$$\left\{ \begin{array}{l} \dot{x}_1 = y_1, \\ \dot{x}_2 = y_2, \\ \dot{y}_1 = -\omega_1^2 x_1, \\ \dot{y}_2 = -\omega_2^2 x_2, \end{array} \right. \quad \Rightarrow \quad \left\{ \begin{array}{l} \ddot{x}_1 + \omega_1^2 x_1 = 0, \\ \ddot{x}_2 + \omega_2^2 x_2 = 0. \end{array} \right.$$

Example: two coupled oscillators

Question: Is there a relation between unstable eigenvalues of $JH''(u_0)$ and negative eigenvalues of $H''(u_0)$?

Consider energy

$$H = \frac{1}{2}(y_1^2 + y_2^2) + \frac{1}{2}(\omega_1^2 x_1^2 - \lambda_2^2 x_2^2)$$

The quadratic form for H has three positive and one negative eigenvalues.

One of the two oscillators is unstable:

$$\begin{cases} \dot{x}_1 = y_1, \\ \dot{x}_2 = y_2, \\ \dot{y}_1 = -\omega_1^2 x_1, \\ \dot{y}_2 = \lambda_2^2 x_2, \end{cases} \quad \Rightarrow \quad \begin{cases} \ddot{x}_1 + \omega_1^2 x_1 = 0, \\ \ddot{x}_2 - \lambda_2^2 x_2 = 0. \end{cases}$$

Negative index count:

$$N_{\mathrm{re}}(JH) = 1 = N_{\mathrm{neg}}(H)$$

Example: two coupled oscillators

Question: Is there a relation between unstable eigenvalues of $JH''(u_0)$ and negative eigenvalues of $H''(u_0)$?

Consider energy

$$H = \frac{1}{2}(y_1^2 + y_2^2) + \frac{1}{2}(-\lambda_1^2 x_1^2 - \lambda_2^2 x_2^2)$$

The quadratic form for H has two positive and two negative eigenvalues.

Both oscillators are unstable:

$$\begin{cases} \dot{x}_1 = y_1, \\ \dot{x}_2 = y_2, \\ \dot{y}_1 = \lambda_1^2 x_1, \\ \dot{y}_2 = \lambda_2^2 x_2, \end{cases} \quad \Rightarrow \quad \begin{cases} \ddot{x}_1 - \lambda_1^2 x_1 = 0, \\ \ddot{x}_2 - \lambda_2^2 x_2 = 0. \end{cases}$$

Negative index count:

$$N_{\rm re}(JH) = 2 = N_{\rm neg}(H)$$

Example: two coupled oscillators

Question: Is there a relation between unstable eigenvalues of $JH''(u_0)$ and negative eigenvalues of $H''(u_0)$?

Consider energy

$$H = \frac{1}{2}(y_1^2 - y_2^2) + \frac{1}{2}(\omega_1^2 x_1^2 - \omega_2^2 x_2^2)$$

The quadratic form for H has two positive and two negative eigenvalues.

The two oscillators are nevertheless stable:

$$\begin{cases} \dot{x}_1 = y_1, \\ \dot{x}_2 = -y_2, \\ \dot{y}_1 = -\omega_1^2 x_1, \\ \dot{y}_2 = \omega_2^2 x_2, \end{cases} \quad \Rightarrow \quad \begin{cases} \ddot{x}_1 + \omega_1^2 x_1 = 0, \\ \ddot{x}_2 + \omega_2^2 x_2 = 0. \end{cases}$$

Negative index count:

$$2N_{\rm im}^-(JH) = 2 = N_{\rm neg}(H)$$

Example: two coupled oscillators

Question: Is there a relation between unstable eigenvalues of $JH''(u_0)$ and negative eigenvalues of $H''(u_0)$?

Consider energy

$$H = \frac{1}{2}(y_1^2 - y_2^2) + \omega^2 x_1 x_2$$

The quadratic form for H has two positive and two negative eigenvalues.

The two oscillators are unstable with a quartet of complex eigenvalues:

$$\begin{cases} \dot{x}_1 = y_1, \\ \dot{x}_2 = -y_2, \\ \dot{y}_1 = -\omega^2 x_2, \\ \dot{y}_2 = -\omega^2 x_1, \end{cases} \quad \Rightarrow \quad \begin{cases} \ddot{x}_1 + \omega^2 x_2 = 0, \\ \ddot{x}_2 - \omega^2 x_1 = 0, \end{cases} \quad \Rightarrow \quad x_1^{(4)} + \omega^4 x_1 = 0.$$

Negative index count:

$$2N_c(JH) = 2 = N_{\rm neg}(H)$$

Spectral stability theorems

Consider the spectral stability problem:

$$JH''(u_0)v = \lambda v, \quad v \in X.$$

For simplicity, assume a zero-dimensional kernel of $H''(u_0)$.

- **Grillakis, Shatah, Strauss, 1990** Orbital Stability Theory:
  - If $H''(u_0)$ has no negative eigenvalues, then $JH''(u_0)$ has no unstable eigenvalues.
  - If $H''(u_0)$ has an odd number of negative eigenvalues, then $JH''(u_0)$ has at least one real unstable eigenvalue.
- **Kapitula, Kevrekidis, Sandstede, 2004**; **Pelinovsky, 2005** Negative Index Theory:

$$N_{\rm re}(JH''(u_0)) + 2N_{\rm c}(JH''(u_0)) + 2N_{\rm im}^{-}(JH''(u_0)) = N_{\rm neg}(H''(u_0)).$$

What is Krein signature for eigenvalues?

- Suppose that $\lambda \in i\mathbb{R}$ is a simple isolated eigenvalue of $JH''(u_0)$ with the eigenvector v. Then, the sign of

$$E''_\omega(v) = \langle H''(u_0)v, v\rangle_{L^2}$$

is called the Krein signature of the eigenvalue λ.

- If λ is a multiple isolated eigenvalue of $JH''(u_0)$, then the number $N^-_{\rm im}(JH''(u_0))$ is introduced as the number of negative eigenvalues of the quadratic form $E''_\omega(v)$ restricted at the invariant subspace of $JH''(u_0)$ associated with the eigenvalue λ.

Example of two coupled NLS equations

Consider the system of two coupled NLS equations:

$$\begin{aligned} iu_t + u_{xx} + \left(|u|^2 + \chi|v|^2\right) u &= 0, \\ iv_t + v_{xx} + \left(\chi|u|^2 + |v|^2\right) v &= 0, \end{aligned}$$

where $\chi > 0$ is the coupling constant.

Stationary states are given by

$$u = U(x)e^{it}, \quad v = V(x)e^{i\omega t},$$

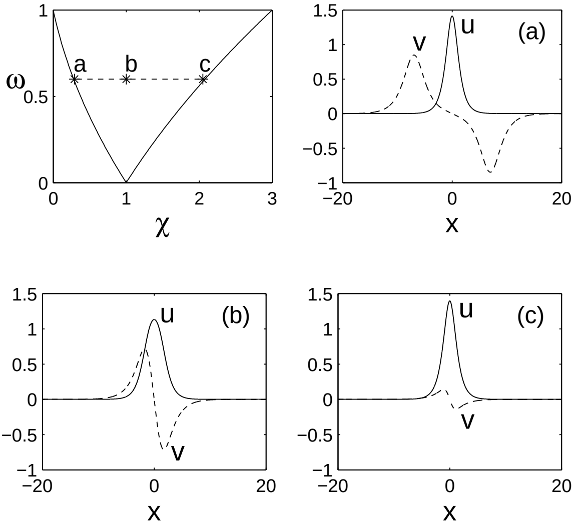
where $\omega > 0$ is a frequency parameter.

Consider families of excited states, for which $U(x) > 0$ for all $x \in \mathbb{R}$ and

$$V(x) \text{ has } n \text{ zeros on } \mathbb{R}.$$

Reference: D.P., J. Yang, Stud. Appl. Math. **115** (2005), 109–137.

Example $n = 1$



Count of eigenvalues

The ODE system for stationary states:

$$\begin{aligned} U'' - U + \left(U^2 + \chi V^2\right) U &= 0, \\ V'' - \omega V + \left(\chi U^2 + V^2\right) v &= 0. \end{aligned}$$

- If V is small [near point (c)], then

  $$U(x) = \sqrt{2}\,\mathrm{sech} x + \mathcal{O}(\epsilon^2), \quad V(x) = \epsilon \phi_n(x) + \mathcal{O}(\epsilon^3), \quad \omega = \omega_n + \mathcal{O}(\epsilon^2),$$

  where (ω_n, ϕ_n) is an eigenvalue–eigenfunction pair of

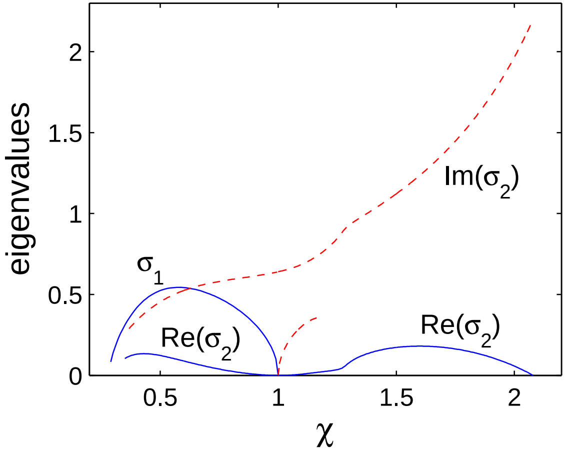
  $$\left(-\frac{d^2}{dx^2} + \omega_n - 2\chi\,\mathrm{sech}^2(x)\right)\phi_n = 0.$$

  where ϕ_n has n zeros on $\mathbb{R}$.

- For small positive ϵ, the negative index count

  $$N_{\rm re}(JH''(u_0)) + 2N_{\rm im}^-(JD''H(u_0)) + 2N_{\rm c}(JH''(u_0)) = 2n,$$

Example $n = 1$



For $1 < \chi < \chi_2$, we have

$$2N_c(JH''(u_0)) = 2 = N_{\rm neg}(H''(u_0)).$$

Why spectral instability?

The spectral stability problem

$$JH''(u_0)v = \lambda v, \quad v \in X.$$

has continuous spectrum for $|\mathrm{Im}(\lambda)| > 1$.

The distance between two consequent eigenvalues,

$$|\omega_n - \omega_{n-1}| > 1, \quad n \in \mathbb{N},$$

therefore, the eigenvalues of negative Krein signature $\lambda = \pm i|\omega_1 - \omega_0|$ are embedded into continuous spectrum, inducing spectral instability.

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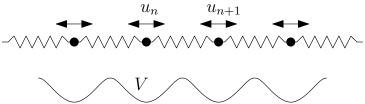
If the eigenvalues of negative Krein signature are isolated, they are spectrally stable. Are these excited states also stable in nonlinear dynamics?

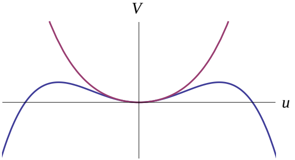
Klein-Gordon lattice

Klein-Gordon (KG) lattice models a chain of coupled anharmonic oscillators with a nearest-neighbour interactions

$$\ddot{u}_n + V'(u_n) = \epsilon(u_{n+1} - 2u_n + u_{n-1}),$$

where $\{u_n(t)\}_{n\in\mathbb{Z}} : \mathbb{R} \to \mathbb{R}^{\mathbb{Z}}$, dot represents time derivative, ϵ is the coupling constant, and $V : \mathbb{R} \to \mathbb{R}$ is an on-site potential.





Applications:

- dislocations in crystals (e.g. Frenkel & Kontorova '1938)
- oscillations in biological molecules (e.g. Peyrard & Bishop '1989)

Nonlinear Schrödinger lattice

Discrete nonlinear Schrodinger equation (dNLS) corresponds to the small-amplitude weakly coupled limit of the KG lattice with $V'(u) = u \pm u^3$:

$$2i\dot{a}_n \pm 3|a_n|^2 a = a_{n+1} - 2a_n + a_{n-1},$$

where $\{a_n(\tau)\}_{n\in\mathbb{Z}} : \mathbb{R} \to \mathbb{C}^{\mathbb{Z}}$ and τ is new time variable.

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By using the leading-order approximation

$$U_j(t) = \epsilon^{1/2}\left[a_j(\epsilon t)e^{it} + \bar{a}_j(\epsilon t)e^{-it}\right],$$

in dKG, one can obtain dNLS and estimate the residual terms

$$\mathrm{Res}_j(t) := \pm\epsilon^{3/2}\left(a_j^3 e^{3it} + \bar{a}_j^3 e^{-3it}\right) + \epsilon^{5/2}\left(\ddot{a}_j e^{it} + \ddot{\bar{a}}_j e^{-it}\right),$$

For every $|t| \leq \tau_0\epsilon^{-1}$, there is $C > 0$ such that

$$\|\mathbf{u}(t) - \mathbf{U}(t)\|_{l^2} + \|\dot{\mathbf{u}}(t) - \dot{\mathbf{U}}(t)\|_{l^2} \leq C\epsilon^{3/2}.$$

Reference: D.P., T. Penati, S. Paleari (2016).

Individual oscillators

In the anti-continuum limit ($\epsilon = 0$), each oscillator is governed by

$$\ddot{\varphi} + V'(\varphi) = 0, \quad \Rightarrow \quad \frac{1}{2}\dot{\varphi}^2 + V(\varphi) = E,$$

where $\varphi \in H^2_{per}(0, T)$ and $V'(u) = \frac{1}{2}u \pm \frac{1}{4}u^3$.

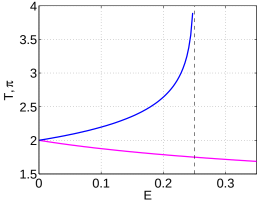


Figure: Period vs. energy in hard (magenta) and soft (blue) V.

The period of the oscillator is

$$T(E) = \sqrt{2} \int_{-a(E)}^{a(E)} \frac{dx}{\sqrt{E - V(x)}},$$

where $a(E)$, the amplitude, is the smallest root of $V(a) = E$.

Multi-breathers near the anti-continuum limit

Breathers are spatially localized time-periodic solutions to the Klein-Gordon lattice. Multi-breathers are constructed by parameter continuation in ϵ from $\epsilon = 0$ and the limiting configuration:

$$\mathbf{u}^{(0)}(t) = \sum_{k \in S} \sigma_k \varphi(t) \mathbf{e}_k \quad \in \quad H^2_{per}((0,T); l^2(\mathbb{Z})),$$

where $S \subset \mathbb{Z}$ is the set of excited sites and $\mathbf{e}_k$ is the unit vector in $l^2(\mathbb{Z})$ at the node k. The oscillators are in-phase if $\sigma_k = +1$ and anti-phase if $\sigma_k = -1$.

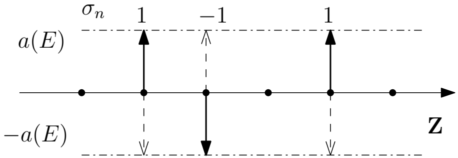


Figure: An example of a multi-site discrete breather at $\epsilon = 0$.

Reference: MacKay & Aubry '1994

Spectral stability of discrete breathers

- Archilla, Cuevas, Sánchez-Rey, Alvarez '2003
- Koukouloyannis, Kevrekidis '2009
- Pelinovsky, Sakovich '2012
- Youshimura '2012

Short summary of stability results:

- Single-site breather - spectrally stable
- Two-site breathers at two adjacent sites:
  - spectrally unstable if in-phase (soft) or anti-phase (hard)
  - spectrally stable if anti-phase (soft) or in-phase (hard)

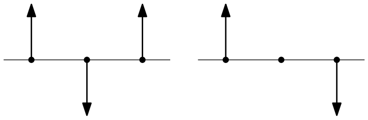
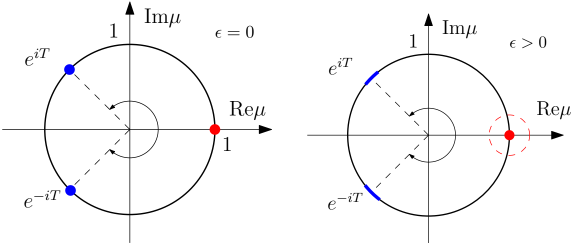


Figure: Stable configuration in soft potential: $T'(E) > 0$.

Spectral stability via Floquet multipliers

For $\epsilon > 0$, Floquet multipliers split as follows:



Two-site breathers have one split pair of Floquet multipliers:

- the pair is on the unit circle if the breathers are spectrally stable
- the pair is on the real line if the breathers are unstable

Question: Are spectrally stable two-site breathers also nonlinearly stable?

Nonlinear instability in dNLS

Consider the NLS (continuous or discrete):

$$i\partial_t u = -\Delta u + V(x)u + |u|^2 u,$$

where V is an external smooth potential with a fast decay at infinity.

Assumptions:

- The Schrödinger operator $-\Delta + V$ admits two simple eigenvalues $E_0 < E_1 < 0$ satisfying

$$|E_1 - E_0| < |E_1|, \quad 2|E_1 - E_0| > |E_1|.$$

- Fermi golden rule of coupling between $2(E_1 - E_0) > |E_1|$ and the continuous spectrum of $-\Delta + V - E_1$ is nonzero.

The excited state bifurcating from E_1 is spectrally stable but nonlinearly unstable.

Cuccagna (2009); Cuccagna–Maeda (2013); Kevrekidis–P.–Saxena (2015).

Spectral stability

Linearizing at the standing wave solution with

$$u(x,t) = e^{-i\omega t}\left[\phi(x) + \delta\left(a(x)e^{\lambda t} + \bar{b}(x)e^{\bar{\lambda}t}\right)\right],$$

we obtain the spectral problem

$$H''(\phi)\psi = i\lambda\sigma_3\psi,$$

where $\psi = (a,b)^T$, $\sigma_3 = \mathrm{diag}(1,-1)$, and H is given by

$$H''(\phi) = \begin{bmatrix} -\Delta + V - \omega + 2|\phi|^2 & \phi^2 \\ \bar{\phi}^2 & -\Delta + V - \omega + 2|\phi|^2 \end{bmatrix}.$$

- the continuous spectrum $\lambda \in i(-\infty, -|\omega|]$ and $\lambda \in i[|\omega|, \infty)$
- the double zero eigenvalue at $\lambda = 0$ due to the gauge symmetry
- the pair of internal modes at $\lambda = \pm i\Omega$, with $\Omega = E_1 - E_0 + \mathcal{O}(\epsilon^2) > 0$.

If $\Omega < |\omega|$, the standing wave is spectrally stable.

Krein quantity

However, the negative index count is $n(H''(\phi)) = 2$.

For eigenvalue $\lambda = i\Omega$ with $\Omega \in \mathbb{R}$, the Krein quantity expresses the energy

$$K = \langle H\psi_\Omega, \psi_\Omega \rangle = -\Omega \int_{\mathbb{R}} (|a_\Omega|^2 - |b_\Omega|^2) dx,$$

where $\psi_\Omega = (a_\Omega, b_\Omega)^T$ is the corresponding eigenvector.

- For continuous spectrum with $\Omega > |\omega|$, $a_\Omega = \mathcal{O}(\epsilon^2)$ and $K > 0$.
- For the internal mode with $\Omega = E_1 - E_0 + \mathcal{O}(\epsilon^2) > 0$, $b_\Omega = \mathcal{O}(\epsilon^2)$ and $K < 0$.

If $\Omega < |\omega|$ but $2\Omega > |\omega|$, then the internal mode eigenfrequency is isolated from the continuous spectrum but the double frequency is embedded into the continuous spectrum.

Nonlinear instability

Using the asymptotic multi-scale expansion

$$u(x,t) = e^{-i\omega t}\left[\phi(x) + \delta\left(c(\tau)a_\Omega(x)e^{i\Omega t} + \bar{c}(\tau)\bar{b}_\Omega(x)e^{-i\Omega t}\right) + \mathcal{O}(\delta^2)\right],$$

computing radiation projections at $\mathcal{O}(\delta^2)$,

$$(H + 2\Omega\sigma_3)\psi_2 = -\phi\left[\begin{array}{c}(a_\Omega + 2b_\Omega)a_\Omega \\ (2a_\Omega + b_\Omega)b_\Omega\end{array}\right],$$

and removing secular terms at $\mathcal{O}(\delta^3)$, we obtain

$$iK\frac{dc}{d\tau} + \Omega\beta|c|^2 c = 0,$$

where $K < 0$ is the Krein quantity at the internal mode,
and $\mathrm{Im}(\beta) > 0$ due to Fermi Golden Rule.

Thus, $|c|^2$ grows in τ, the standing wave is nonlinearly unstable.

Numerical illustration in 1D

The dNLS equation

$$i\dot{u}_n + C(u_{n+1} - 2u_n + u_{n-1}) + |u_n|^2 u_n = 0, \quad n \in \mathbb{Z}.$$

For $C = 0.07$ and $\omega = 1$, we have $\Omega \approx 0.598$, so that $\Omega < \omega$ but $2\Omega > \omega$.

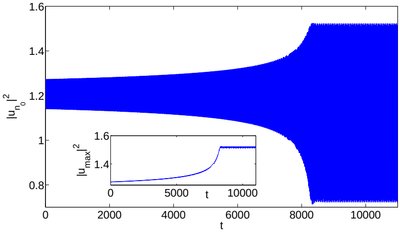


Figure: Evolution of a two-site localized mode in 1D dNLS. A transition to a quasi-periodic localized mode is observed.

Numerical illustration in 2D

The dNLS equation

$$i\dot{u}_{n,m} + C(\Delta u)_{n,m} + \frac{|u_{n,m}|^2 u_{n,m}}{1 + |u_{n,m}|^2} = 0, \quad (n,m) \in \mathbb{Z}^2.$$

For $C = 0.09$ and $\omega = 0.35$, we have $\Omega_1 \approx 0.19$, $\Omega_2 \approx 0.16$, and $\Omega_3 \approx 0.01$, so that $\Omega_3 < \Omega_2 < \Omega_1 < \omega$ but $2\Omega_1 > \omega$.

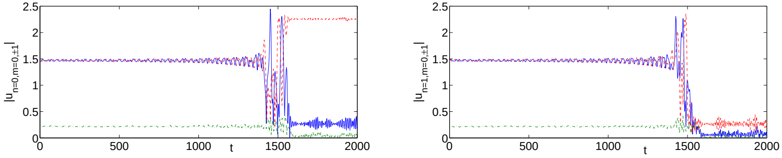


Figure: Evolution of a vortex residing on four sites of the 2D dNLS. A relaxation to a single-site soliton is observed.

Nonlinear instability in dKG

Consider the discrete KG equation

$$\ddot{u}_n + V'(u_n) = \varepsilon(u_{n+1} - 2u_n + u_{n-1}), \quad n \in \mathbb{Z},$$

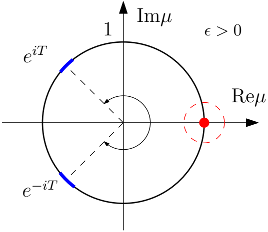
where V is smooth and $V = \frac{1}{2}u^2 + \mathcal{O}(u^3)$.

Assumptions: Similar to the NLS case, but more restrictive for the second-order equations...

If an internal mode has Krein signature opposite to that of the spectral band, then the breather is nonlinearly unstable.

J. Cuevas, P.G. Kevrekidis, D.P. (2016).

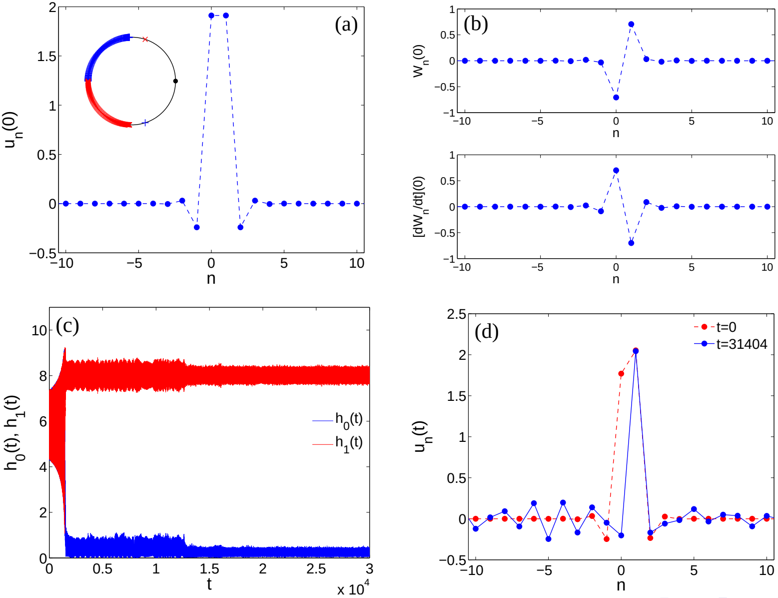
Krein quantity



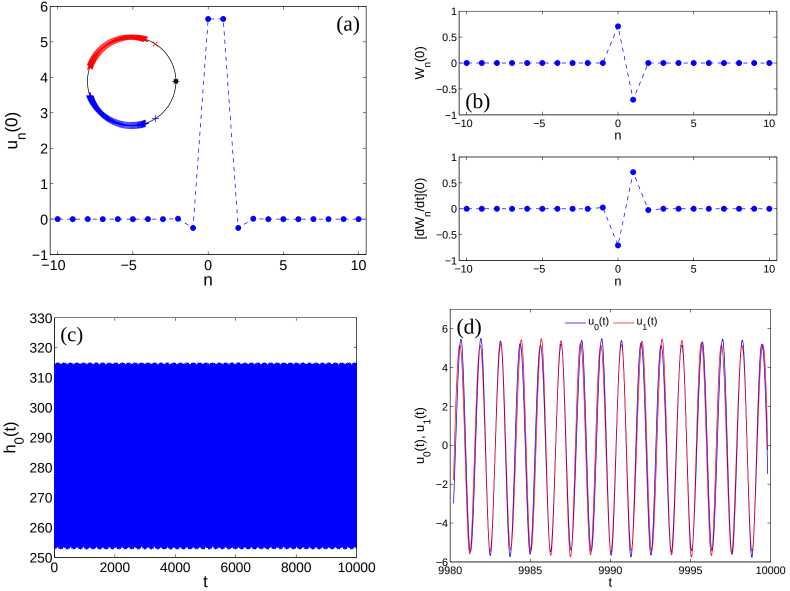
For the hard potential with $T'(E) < 0$ and $T(E) < 2\pi$,

- $0 < T < \pi$: the Krein signatures of the internal mode and the wave spectrum in the upper semi-circle coincide;
- $\pi < T < 2\pi$: the Krein signatures of the internal mode and the wave spectrum in the upper semi-circle are opposite to each other.

Numerical illustration: hard ϕ^4 potential $T = \pi$



Numerical illustration: hard ϕ^4 potential $T < \pi$



Conclusions

- Excited localized modes occur everywhere in Hamiltonian systems: nodal solitons, multi-site breathers, vortices...
- Negative eigenvalues of quadratic Hamiltonian show up in the spectral stability problem either as unstable eigenvalues or as stable eigenvalues of negative Krein signature.
- In the latter case, nonlinear instabilities may destroy localized modes in spite of their spectral stability.
- Spectrally stable multi-site breathers in lattices are either nonlinearly stable or unstable, depending on the breather period T.

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Thank you.