Dynamics of shocks in the modular Burgers equation

Dmitry Pelinovsky

Department of Mathematics, McMaster University

with

Uyen Le, Jeanne Lin - McMaster University

Pascal Poullet - Universite de Antilles

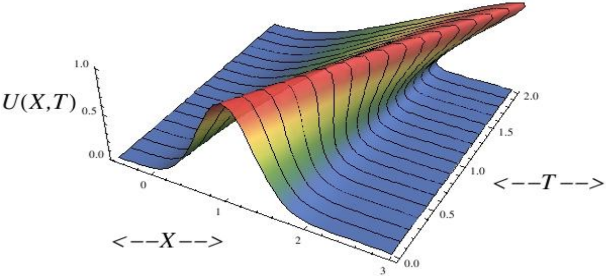
Bjorn de Rijk - Kalrsruhe Institute of Technology

Inviscid Shocks

- Dynamics of a Conservation Law

$$\partial_t v + \partial_x f(v) = 0$$

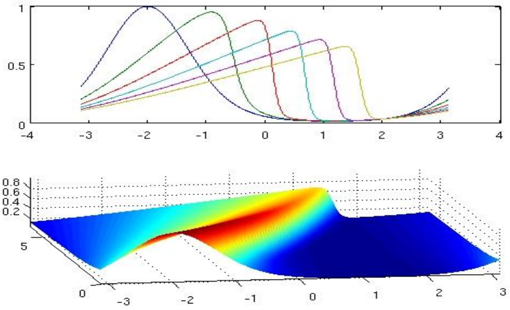
generate shock singularities in finite time from a large class of smooth data and for smooth $f(v)$.



Viscous Shocks

- Diffusive regularization is modeled by a viscous Burgers equation

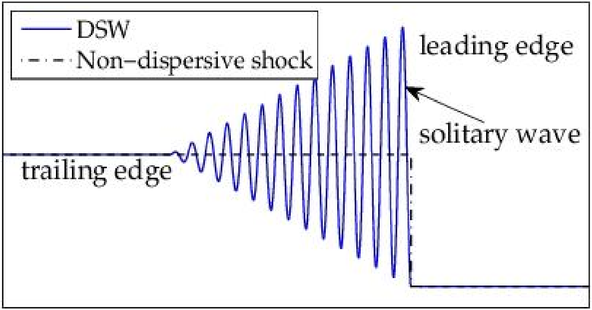
$$\partial_t v + \partial_x f(v) = \varepsilon^2 \partial_x^2 v.$$



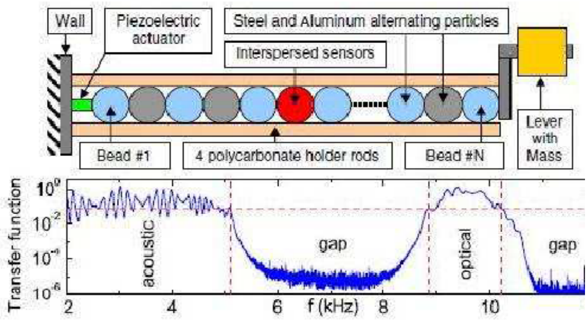
Dispersive Shocks

- Dispersive regularization is modeled by the KdV equation

$$\partial_t v + \partial_x f(v) + \varepsilon^3 \partial_x^3 v = 0.$$



Granular chains



- Granular chains contain densely packed, elastically interacting particles with Hertzian contact forces.

V. Nesterenko, C. Daraio, P.G. Kevrekidis, G. Theocharis, and many more.

Logarithmic models

Granular chains are modeled with Newton's equations of motion:

$$x_n''(t) = V'(x_{n+1} - x_n) - V'(x_n - x_{n-1}), \quad n \in \mathbb{Z},$$

where x_n is the displacement of the nth particle and V is the interaction potential for spherical beads (H. Hertz, 1882):

$$V(x) = |x|^{1+\alpha} H(-x), \quad \alpha = \frac{3}{2},$$

where H is the step (Heaviside) function. For hollow materials, $\alpha \to 1$.

- The conservative model yields the logarithmic KdV equation

$$\partial_t v + \partial_x (v \log |v|) + \partial_x^3 v = 0$$

- The dissipative model yields the logarthmic Burgers equation

$$\partial_t v + \partial_x (v \log |v|) = \partial_x^2 v$$

G. James & D. P., 2014; G. James, 2021

Modular nonlinearity

In a similar context of dynamics of particles with piecewise interaction potentials, models with modular nonlinearities have been derived:

- The modular KdV equation

$$\partial_t v = \partial_x |v| + \partial_x^3 v$$

- The modular Burgers equation

$$\partial_t v = \partial_x |v| + \partial_x^2 v$$

C. M. Hedberg, O. V. Rudenko, 2016–2018

The models are linear for sign-definite solutions. Nonlinear waves correspond to the sign-changing solutions, for which the modeling problem becomes a moving interface problem between solutions of linear equations.

Traveling waves in the modular Burgers equation

Starting with

$$\partial_t v = \partial_x |v| + \partial_x^2 v,$$

we can think of the traveling wave solutions $v(t,x) = W(x - ct)$, where

$$W''(x) + \mathrm{sign}(W) W'(x) + cW'(x) = 0, \quad x \in \mathbb{R}.$$

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Q What is the function space for solutions?

A Space of piecewise C^2 functions satisfying the interface conditiion

$$[W'']_-^+(x_0) = -2|W'(x_0)|$$

at each interface located at x_0, where $[f]_-^+(x_0) = f(x_0^+) - f(x_0^-)$ is the jump of a piecewise continuous function f across x_0.

Traveling waves in the modular Burgers equation

Integrating once yields

$$W'(x) + |W(x)| + cW(x) = d, \quad x \in \mathbb{R},$$

where the constant of integration is identical for all pieces of piecewise C^2 function $W(x) : \mathbb{R} \to \mathbb{R}$.

Traveling waves in the modular Burgers equation

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where the constant of integration is identical for all pieces of piecewise C^2 function $W(x) : \mathbb{R} \to \mathbb{R}$.

If $W_\pm = \lim_{x \to \pm\infty} W(x)$, then bounded solutions only exist if and only if $W_- < 0 < W_+$ with uniquely selected speed

$$c = \frac{W_+ + W_-}{W_+ - W_-}$$

and uniquely defined profile W up to spatial translations:

$$W(x) = \begin{cases} W_+(1 - e^{-(1+c)x}), & x > 0, \\ W_-(1 - e^{(1-c)x}), & x < 0. \end{cases}$$

If $W_+ = -W_-$, then $c = 0$ and $W(-x) = -W(x)$ is odd.

Motivations

1. Is the viscous shock W stable in the time evolution of the modular Burgers equation?
2. How does the interface moves in the time evolution depending on the initial conditions?
3. Is there the finite-time extinction of the area between two consequent interfaces?
4. How can we model the moving interface problems numerically?

Interface equation

It is natural to look for solutions of the modular Burgers equation

$$\begin{cases} \partial_t v = \partial_x |v| + \partial_x^2 v, & t > 0, \quad x \in \mathbb{R}, \\ v(0, \cdot) = v_0 \end{cases}$$

in class of piecewise C^2 functions of $x \in \mathbb{R}$ for every $t \geq 0$.

If $v(t, \xi(t)) = 0$ defines the interface at $x = \xi(t)$, then

$$[v_t]_-^+(\xi(t)) = 0 \quad \text{and} \quad [v_x]_-^+(\xi(t)) = 0,$$

whereas

$$[v_{xx}]_-^+(\xi(t)) = -2|v_x(t, \xi(t))|$$

determines the interface equation for $\xi(t)$, $t > 0$.

Simple case: odd data

It follows from

$$\partial_t v = \partial_x |v| + \partial_x^2 v$$

that if $v(0,-x) = -v(0,x)$ is odd at $t = 0$, then $v(t,-x) = -v(t,x)$ remains odd for all $t > 0$. One interface is located at $\xi(t) = 0$, $t > 0$.

Adding an odd perturbation $w(t,x)$ to the odd viscous shock $W(x) = (1 - e^{-|x|})\mathrm{sgn}(x)$ with $c = 0$ as $v(t,x) = W(x) + w(t,x)$, we get the linear initial-boundary-value problem

$$\begin{cases} w_t = w_x + w_{xx}, & x > 0, & t > 0, \\ w(t,0) = 0, & & t > 0, \\ w(t,x) \to 0 & \text{as } \; x \to +\infty, & t > 0, \\ w(0,x) = w_0(x), & x > 0, & \end{cases}$$

Asymptotic stability: odd data

Theorem (Le, P., Poullet, 2021)

For every $\epsilon > 0$ *there is* $\delta > 0$ *such that for every odd* v_0 *satisfying*

$$\|v_0 - W\|_{H^2} < \delta,$$

there exists a unique odd solution $v(t,x)$ *with* $v(0,x) = v_0(x)$ *satisfying*

$$\|v(t,\cdot) - W\|_{H^2} < \epsilon, \quad t > 0$$

and

$$\|v(t,\cdot) - W\|_{L^\infty} \to 0 \quad \text{as} \quad t \to +\infty.$$

- Since $W(0) = 0$, $W'(0) = 1$, and H^2 is embedded into C^1, we have $v(t,x) = W(x) + w(t,x) > 0$ for every $x > 0$ and $t > 0$.

General case: single interface

Consider the viscous shock $W(x) = (1 - e^{-|x|})\mathrm{sgn}(x)$ with $c = 0$ but make no assumption on the symmetry of perturbations. With the decomposition

$$v(t,x) = W(x - \xi(t)) + w(t, x - \xi(t)), \quad y = x - \xi(t),$$

we have now the linear initial-boundary-value problem

$$\begin{cases} w_t = (\xi'(t) \pm 1) w_y + w_{yy} + \xi'(t) W'(y), & \pm y > 0, & t > 0, \\ w(t,0) = 0, & & t > 0, \\ w(t,x) \to 0 & \text{as } \ y \to \pm\infty, & t > 0, \\ w(0,y) = w_0(y), & y \in \mathbb{R}, & \end{cases}$$

The two equations on half-lines are coupled by the interface conditions

$$(\xi'(t) \pm 1) w_y(t, 0^{\pm}) + w_{yy}(t, 0^{\pm}) + \xi'(t) = 0,$$

which are consistent due to the conditions $[w_{xx}]_-^+(\xi(t)) = -2|w_x(t, \xi(t))|$.

Asymptotic stability: general data

Theorem (Le, P., Poullet, 2021)

Fix $\alpha \in \left(0, \frac{1}{2}\right)$. *For every* $\epsilon > 0$ *there is* $\delta > 0$ *s.t. for every* v_0 *s.t.*

$$\|v_0 - W\|_{H^2} + \|e^{\alpha|\cdot|}(v_0 - W)\|_{W^{2,\infty}} < \delta$$

there exists a unique solution $v(t,x)$ *with* $v(0,x) = v_0(x)$ *satisfying*

$$\|v(t, \cdot + \xi(t)) - W\|_{H^2} + \|e^{\alpha|\cdot|}(v(t, \cdot + \xi(t)) - W)\|_{W^{2,\infty}} < \epsilon, \quad t > 0$$

and

$$\|v(t, \cdot + \xi(t)) - W\|_{L^\infty} \to 0 \quad \text{as} \quad t \to +\infty,$$

with $\xi' \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$.

Asymptotic stability: general data

- First step: for a given class of $\xi' \in L^1(\mathbb{R}_+) \cup L^\infty(\mathbb{R}_+)$, solve the two boundary-value problems for $w\pm(t,\cdot)$ with $\pm y > 0$. The two solutions are uncoupled.
- Second step: impose the condition $w_y^+(t,0) = w_y^-(t,0)$ as an integral equation on $\xi' \in L^1(\mathbb{R}_+) \cup L^\infty(\mathbb{R}_+)$. This equation can be uniquely solved by using Abel's integral equations.
- Since $\xi' \in L^1(\mathbb{R}_+)$, there exists $\xi_\infty := \lim_{t\to\infty} \xi(t)$, which is defined by the initial data u_0.

Reformulation for numerical approximations

The original problem for general perturbation $w(t, y)$ with $y = x - \xi(t)$:

$$\begin{cases} w_t = (\xi'(t) \pm 1) w_y + w_{yy} + \xi'(t) e^{-y}, & \pm y > 0, \quad t > 0, \\ w(t,0) = 0, & t > 0, \\ w(t,x) \to 0 & \text{as} \;\; y \to \pm\infty, \quad t > 0, \\ w(0,y) = w_0(y), & y \in \mathbb{R}, \end{cases}$$

By using variables $v^{\pm}(t,y) := w(t,y) \mp w(t,-y)$ with $y > 0$ we obtain the coupled system

$$\begin{cases} v_t^+ = v_y^+ + v_{yy}^+ + \xi'(t) v_y^-, & y > 0, \\ v_t^- = v_y^- + v_{yy}^- + \xi'(t) v_y^+ + 2\xi'(t) e^{-y}, & y > 0, \end{cases}$$

subject to $v^{\pm}(t,0) = 0$, $v_y^-(t,0) = 0$, and $\xi'(t) = -\frac{v_{yy}^-(t,0)}{2 + v_y^+(t,0)}$.

Remarks on the numerical method

- Central-difference approximation of spatial derivatives.
- Neumann condition for $v_y^-(t,0) = 0$ is modelled with an extra grid point $v_{-1}^-(t) = v_1^-(t)$.
- The smoothness condition for $v_y^+(t,0) + v_{yy}^+(t,0) = 0$ is modelled with an extra grid point

$$v_{-1}^+(t) = -\frac{2+h}{2-h} v_1^+(t).$$

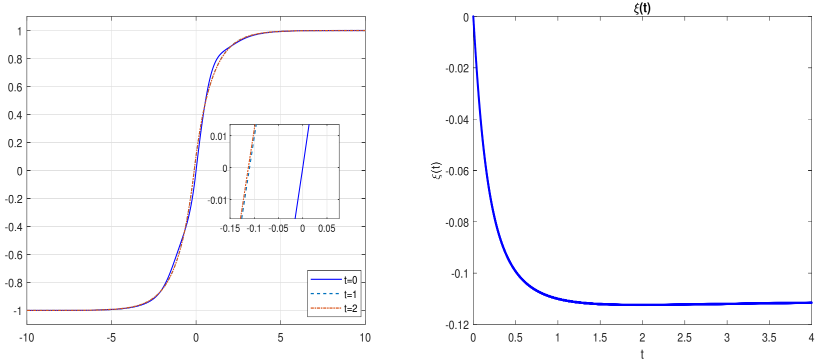
- The interface condition $\xi'(t) = -\frac{v_{yy}^-(t,0)}{2+v_y^+(t,0)}$ is resolved as

$$\xi'(t) = -\frac{(2-h)v_1^-(t)}{hv_1^+(t) + h^2(2-h)}.$$

- Time steps are performed with the implicit Crank-Nicholson method

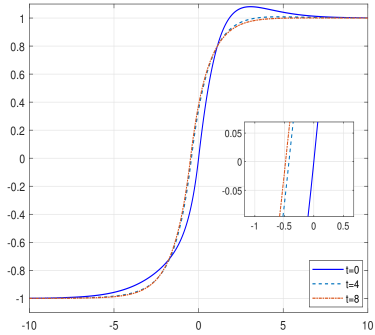
Initial data with Gaussian decay

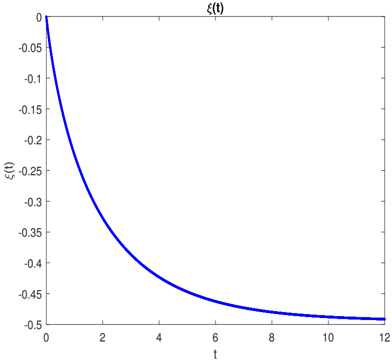
$$v^{+}(0,y) = 0.1(y - 0.5y^2)e^{-y^2}, \quad v^{-}(0,y) = 0.5y^2 e^{-y^2}.$$



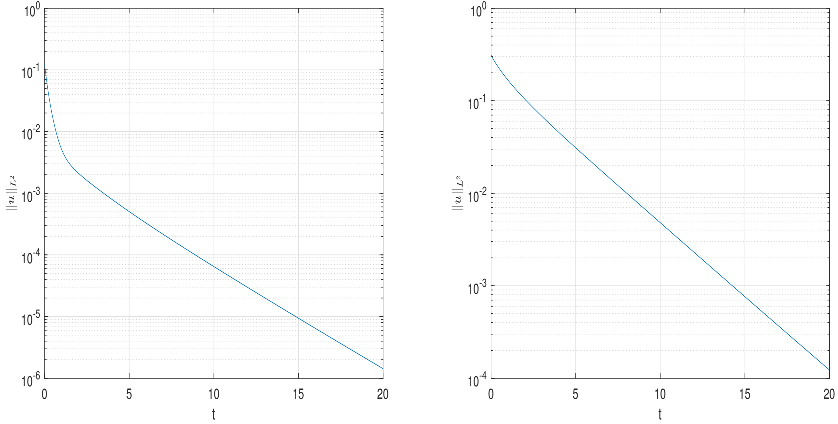
Initial data with exponential decay

$$v^+(0,y) = 0.1(y + 0.5y^2)e^{-y}, \quad v^-(0,y) = 0.5y^2 e^{-y},$$



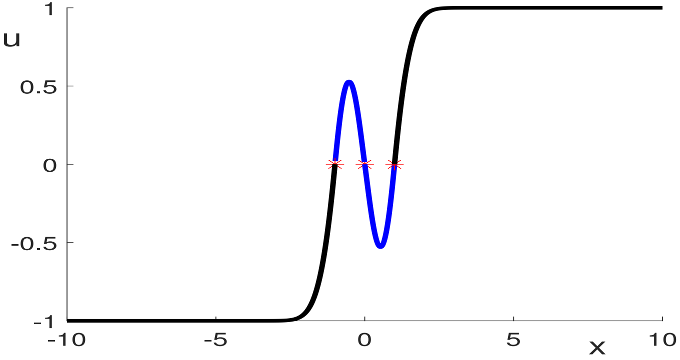


Convergence in time for L^2-norm of perturbation



Initial data with multiple interfaces

Main question: Is there the finite-time extinction of the area between two consequent interfaces for $u_t = (|u|)_x + u_{xx}$?



Interface at $x = 0$ persists for odd data. Interfaces at $x = \pm\xi(t)$ move.

A simple argument suggesting the finite-time coalescence [P., de Rijk, 2023]

Let $z(t,x) := 1 - u(t,x)$. If $z(0,\cdot) : (0,\infty) \to \mathbb{R}$ is positive and integrable, then $z(t,\cdot) : (0,\infty) \to \mathbb{R}$ is positive and integrable for $t > 0$ by comparison principle.

We have for some time $t \in [0, \tau_0)$

$$0 < \xi(t) \leq \int_0^{\xi(t)} z(t,x)dx \leq \int_0^{\infty} z(t,x)dx =: M(t),$$

because $z(t,x) \geq 1$ for $x \in [0, \xi(t)]$ and $z(t,x) \geq 0$ for $x \in [\xi(t), \infty)$.

On the other hand,

$$\frac{dM}{dt} = -1 - z_x(t,0) \leq -1.$$

Hence, $M(t) \leq M(0) - t$ and we have finite-time coalescence: $\xi(\tau_0) = 0$.

Reformulation for numerical approximations

The original problem is

$$\begin{cases} u_t = -u_x + u_{xx}, & u(t,x) < 0, & 0 < x < \xi(t), \\ u_t = u_x + u_{xx}, & u(t,x) > 0, & \xi(t) < x < \infty, \\ u(t,0) = 0, & u(t,\xi(t)) = 0, & \lim_{x \to +\infty} u(t,x) = 1, \end{cases}$$

By using $y := x/\xi(t)$, the boundary-value problem is mapped to the time-independent regions:

$$\begin{cases} u_t = \xi^{-1}(\xi' y - 1)u_y + \xi^{-2} u_{yy}, & u(t,y) < 0, & 0 < y < 1, \\ u_t = \xi^{-1}(\xi' y + 1)u_y + \xi^{-2} u_{yy}, & u(t,y) > 0, & 1 < y < \infty, \\ u(t,0) = 0, & u(t,1) = 0, & \lim_{y \to +\infty} u(t,y) = 1, \end{cases}$$

closed with the interface condition:

$$\xi'(t) = -1 - \frac{u_{yy}(t,1^+)}{\xi(t) u_y(t,1)} = +1 - \frac{u_{yy}(t,1^-)}{\xi(t) u_y(t,1)}.$$

Remarks on the numerical method

- Central-difference approximation of spatial derivatives.
- The grid on $[0,1]$ is complemented with the extra grid point $y_{N+1} = 1 + h$ and the approximation u^*_{N+1}. The grid on $[1,L]$ with $L = 10$ is complemented with the extra grid point $y_{N-1} = 1 - h$ and the approximation u^*_{N-1}. Note that $u^*_{N\pm1} \neq u_{N\pm1}$.
- The additional variables u^*_{N+1} and u^*_{N-1} are found from the interface conditions: $[u_y]^+_-(1) = 0$ and $[u_{yy}]^+_-(1) = -2\xi(t)|u_y(t,1)|$. This yields the relation between linear advection-diffusion equation and

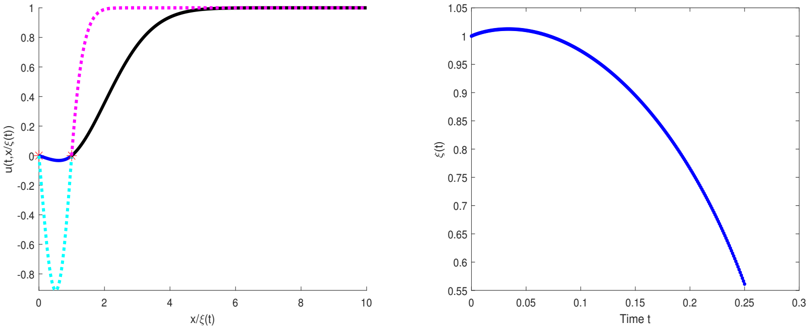
$$\xi'(t) = -\frac{(2 - h\xi)(u_{N+1} + u_{N-1})}{h\xi(u_{N+1} - u_{N-1})}.$$

- Time steps are performed with the implicit Crank-Nicholson method

Initial data and evolution: $\alpha = 1.5$

$$u_0(x) = \begin{cases} x(1-x)(ax^2+bx+c), & 0 < x < 1, \\ 1 - e^{-\alpha(x^2-1)}, & 1 < x < \infty, \end{cases}$$

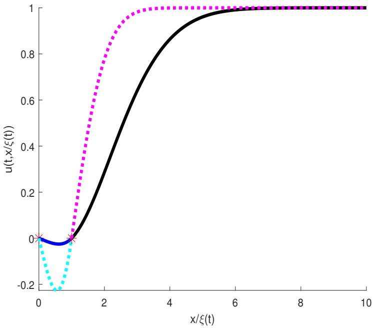
with $\xi'(0) = 2(\alpha - 1)$, where a, b, c are uniquely defined by α.

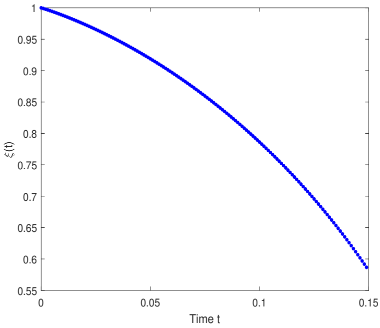


Initial data and evolution: $\alpha = 0.5$

$$u_0(x) = \begin{cases} x(1-x)(ax^2+bx+c), & 0 < x < 1, \\ 1 - e^{-\alpha(x^2-1)}, & 1 < x < \infty, \end{cases}$$

with $\xi'(0) = 2(\alpha - 1)$, where a, b, c are uniquely defined by α.





Conjecture based on numerical data [P., de Rijk, 2023]

There exists $t_0 \in (0, \infty)$ such that

$$\xi(t) \sim \sqrt{t_0 - t}, \quad u_x(t, \xi(t)) \sim (t_0 - t), \quad u_{xx}(t, \xi(t)^-) \sim \sqrt{t_0 - t}.$$

This scaling law is in agreement with

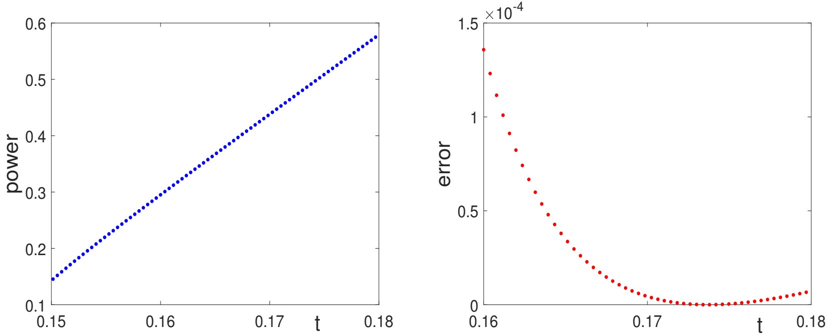
$$\xi'(t) = +1 - \frac{u_{xx}(t, \xi(t)^-)}{u_x(t, 1)}.$$

The method of data extraction, e.g. for $\xi(t) \sim \sqrt{t_0 - t}$

For a fixed value of t_0 (past the termination time of our computations) , we compute c_1 (left) and c_2 in the linear regression

$$\log(\xi(t)) \quad \text{versus} \quad c_1 \log(t_0 - t) + c_2$$

as well as the approximation error (right). The minimal error of 10^{-9} is attained at $t_0 = 0.17$ with $c_1 = 0.492$.



Regularization of the modular nonlinearity

Instead of

$$\partial_t u = \mathrm{sgn}(u)\partial_x u + \partial_x^2 u,$$

we can consider a regularized Burgers equation

$$\partial_t u_\varepsilon = \frac{u_\varepsilon}{\sqrt{\varepsilon^2 + u_\varepsilon^2}}\partial_x u_\varepsilon + \partial_x^2 u_\varepsilon,$$

for very small values of ε.

We considered the initial data among the odd functions:

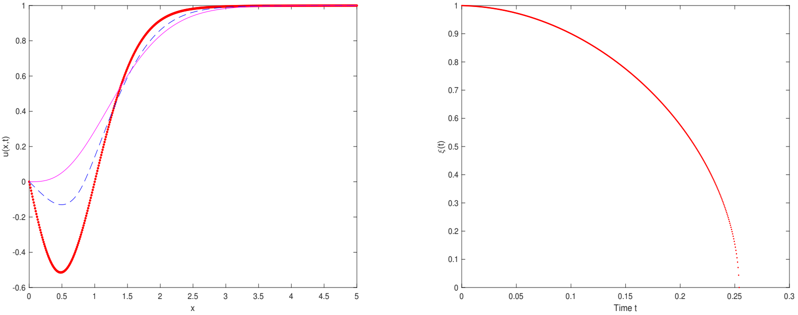
$$\phi(x) = \tanh(x)\left(1 - \frac{\cosh^2(\alpha)}{\cosh^2(\alpha x)}\right)$$

and

$$\phi(x) = \tanh(x)\left(1 - e^{\alpha(1-x^2)}\right),$$

where $\alpha > 0$ is the slope parameter.

Dynamics for $\alpha = 1$

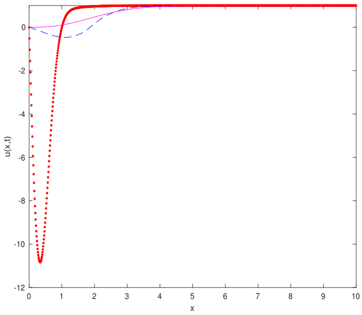


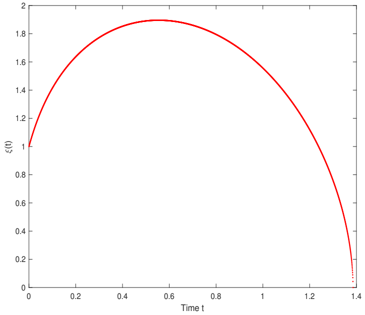
We have confirmed independently of ε:

$$\xi(t) \sim \sqrt{t_0 - t},$$

where $t_0 \approx 0.2538$ and *power* ≈ 0.5068.

Dynamics for $\alpha = 4$





We have confirmed independently of ε:

$$\xi(t) \sim \sqrt{t_0 - t},$$

where $t_0 \approx 1.3853$ and *power* ≈ 0.5127.

A simple argument suggesting the scaling law

Assume that there exists $(t_0, \xi_0) \in \mathbb{R}^+ \times \mathbb{R}^+$ such that

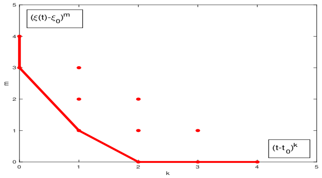
$$u_x(t_0, \xi_0) = 0, \quad u_{xx}(t_0, \xi_0) = 0, \quad \text{and} \quad u_{xxx}(t_0, \xi_0) \neq 0.$$

For smooth nonlinearity, the smooth solution $u \in C^\infty(\mathbb{R}_+ \times \mathbb{R})$ satisfies

$$\begin{aligned}
0 &= u(t, \xi(t)) \\
&= \underbrace{u(t_0, \xi_0)}_{=0} + (t - t_0) \underbrace{u_t(t_0, \xi_0)}_{=0} + (\xi(t) - \xi_0) \underbrace{u_x(t_0, \xi_0)}_{=0} \\
&\quad + \frac{1}{2}(t - t_0)^2 u_{tt}(t_0, \xi_0) + (t - t_0)(\xi(t) - \xi_0) \underbrace{u_{tx}(t_0, \xi_0)}_{\neq 0} + \frac{1}{2}(\xi(t) - \xi_0)^2 \underbrace{u_{xx}(t_0, \xi_0)}_{=0} \\
&\quad + \frac{1}{6}(t - t_0)^3 u_{ttt}(t_0, \xi_0) + \frac{1}{2}(t - t_0)^2(\xi(t) - \xi_0) u_{ttx}(t_0, \xi_0) \\
&\quad + \frac{1}{2}(t - t_0)(\xi(t) - \xi_0)^2 u_{txx}(t_0, \xi_0) + \frac{1}{6}(\xi(t) - \xi_0)^3 \underbrace{u_{xxx}(t_0, \xi_0)}_{\neq 0} + \mathcal{O}(4).
\end{aligned}$$

A simple argument suggesting the scaling law

It follows from the Newton's polygon that there exists a pitchfork bifurcation with two sets of roots of $u(t,\cdot)$ near $x = \xi_0$ for $t \neq t_0$.



One pair of roots disappears at $t = t_0$:

$$\xi_{1,2}(t) - \xi_0 = \pm\sqrt{6(t_0 - t)} + \mathcal{O}(t_0 - t).$$

The third root continus past $t = t_0$:

$$\xi(t) - \xi_0 = \frac{u_{tt}(t_0, \xi_0)}{2u_{xxx}(t_0, \xi_0)}(t_0 - t) + \mathcal{O}((t_0 - t)^2).$$

For odd data, $\xi_0 = 0$ and $\xi(t) = 0$ for all $t > 0$.

Summary

- Evolution of the modular Burgers equation is considered.
- Asymptotic stability of a traveling viscous shock is proven and illustrated numerically.
- It is shown that shock waves with multiple interfaces extinct in a finite time due to finite-time coalesence of interfaces
- A precise scaling law of the finite-time coalescence is suggested based on the numerical data and proved for smooth nonlinearity.

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