Moving gap solitons in periodic potentials

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Motivations

Gap solitons are localized stationary solutions of nonlinear PDEs with space-periodic coefficients which reside in the spectral gaps of associated linear operators.

Examples: Complex-valued Maxwell equation

$$\nabla^2 E - E_{tt} + \left(V(x) + \sigma|E|^2\right) E_{tt} = 0$$

and the Gross–Pitaevskii equation

$$iE_t = -\nabla^2 E + V(x)E + \sigma|E|^2 E,$$

where $E(x,t) : \mathbb{R}^N \times \mathbb{R} \mapsto \mathbb{C}$, $V(x) = V(x + 2\pi e_j) : \mathbb{R}^N \mapsto \mathbb{R}$, and $\sigma = \pm 1$.

Existence of stationary solutions

Stationary solutions $E(x,t) = U(x)e^{-i\omega t}$ with $\omega \in \mathbb{R}$ satisfy a nonlinear elliptic problem with a periodic potential

$$\nabla^2 U + \omega U = V(x)U + \sigma|U|^2 U$$

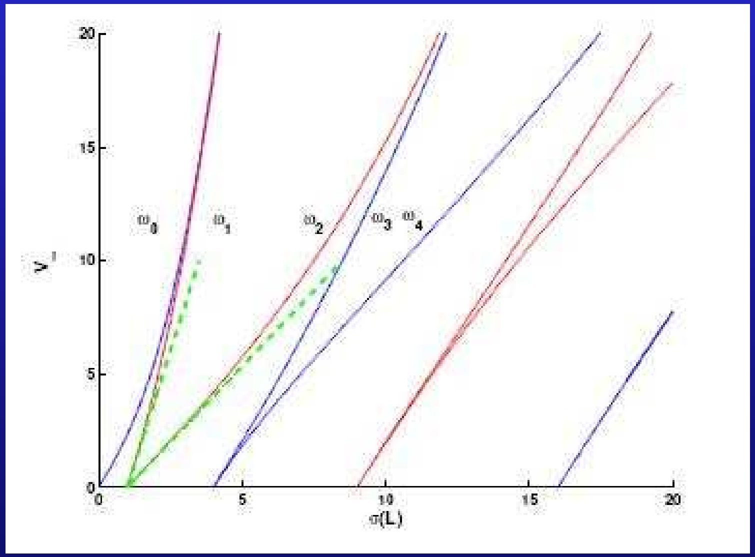
Theorem:[Pankov, 2005] Let $V(x)$ be a real-valued bounded periodic potential. Let ω be in a finite gap of the spectrum of $L = -\nabla^2 + V(x)$. There exists a non-trivial weak solution $U(x) \in H^1(\mathbb{R}^N)$, which is (i) real-valued, (ii) continuous on $x \in \mathbb{R}^N$ and (iii) decays exponentially as $|x| \to \infty$.

Remark: Additionally, there exists a localized solution $U(x) \in H^1(\mathbb{R}^N)$ in the semi-infinite gap for $\sigma = -1$ (NLS soliton).

Coupled-mode theory for gap solitons

Stationary gap solitons can be approximated asymptotically by the coupled-mode theory in one dimension ($N = 1$) in the limit of small-amplitude potentials: $V(x) = \epsilon(1 - \cos x)$ for small ϵ.

The finite-band spectrum of $L = -\partial_x^2 + V(x)$ is shown here:



Coupled-mode equations are derived with asymptotic multi-scale expansions:

$$E(x,t) = \sqrt{\epsilon}\left[a(\epsilon x, \epsilon t)e^{\frac{ix}{2}} + b(\epsilon x, \epsilon t)e^{-\frac{ix}{2}} + \mathrm{O}(\epsilon)\right]e^{-\frac{it}{4}}.$$

Gap solitons in coupled-mode equations

The vector $(a,b) : \mathbb{R} \times \mathbb{R} \mapsto \mathbb{C}^2$ satisfies asymptotically the coupled-mode system:

$$\left\{ \begin{array}{r} i(a_T + a_X) + V_2 b = \sigma(|a|^2 + 2|b|^2)a, \\ i(b_T - b_X) + V_{-2} a = \sigma(2|a|^2 + |b|^2)b, \end{array} \right.$$

where $X = \epsilon x$, $T = \epsilon t$, and $V_2 = \bar{V}_{-2}$ are Fourier coefficients of $V(x)$. Stationary gap solitons are obtained in the analytic form

$$a(X,T) = a(X)e^{-i\Omega T}, \quad b(X,T) = b(X)e^{-i\Omega T},$$

$$a(X) = \bar{b}(X) = \frac{\sqrt{2}}{\sqrt{3}} \frac{\sqrt{|V_2|^2 - \Omega^2}}{\sqrt{|V_2| - \Omega}\cosh(\kappa X) + i\sqrt{|V_2| + \Omega}\sinh(\kappa X)},$$

where $\kappa = \sqrt{|V_2|^2 - \Omega^2}$ and $|\Omega| < |V_2|$.

Moving gap solitons

Moving gap solitons are obtained in the analytic form

$$a = \left(\frac{1+c}{1-c}\right)^{1/4} A(\xi)e^{-i\mu\tau},\ b = \left(\frac{1-c}{1+c}\right)^{1/4} B(\xi)e^{-i\mu\tau},\ |c| < 1,$$

where

$$\xi = \frac{X - cT}{\sqrt{1-c^2}}, \quad \tau = \frac{T - cX}{\sqrt{1-c^2}}$$

and, since $|A|^2 - |B|^2$ is constant in $\xi \in \mathbb{R}$, then

$$A = \phi(\xi)e^{i\varphi(\xi)}, \qquad B = \bar{\phi}(\xi)e^{i\varphi(\xi)},$$

with ϕ and φ being solutions of the system

$$\varphi' = \frac{-2c\sigma|\phi|^2}{(1-c^2)}, \quad i\phi' = V_2\bar{\phi} - \mu\phi + \sigma\frac{(3-c^2)}{(1-c^2)}|\phi|^2\phi.$$

Questions and Answers

Main Questions: (a) Can we justify the use of the coupled-mode theory to approximate stationary gap solitons?

YES: D.P., G.Schneider, Asymptotic Analysis (2007)

(b) Can we justify the use of the coupled-mode theory to approximate moving gap solitons?

NO: this work

Theorem:[Goodman,Weinstein,Holmes, 2001; Schneider,Uecker, 2001:] Let $(a,b) \in C([0,T_0],H^3(\mathbb{R},\mathbb{C}^2))$ be solutions of the time-dependent coupled-mode system for a fixed $T_0 > 0$. There exists $\epsilon_0, C > 0$ such that for all $\epsilon \in (0,\epsilon_0)$ the Gross–Pitaevskii equation has a local solution $E(x,t)$ and

$$\|E(x,t) - \sqrt{\epsilon}\left[a(\epsilon x,\epsilon t)e^{i(kx-\omega t)} + b(\epsilon x,\epsilon t)e^{i(-kx-\omega t)}\right]\|_{H^1(\mathbb{R})} \leq C\epsilon$$

for some (k,ω) and any $t \in [0,T_0/\epsilon]$.

Assumptions of the main theorem

Assumption: Let $V(x)$ be a smooth 2π-periodic real-valued function with zero mean and symmetry $V(x) = V(-x)$ on $x \in \mathbb{R}$, such that

$$V(x) = \sum_{m \in \mathbb{Z}} V_{2m} e^{imx} : \quad \sum_{m \in \mathbb{Z}} (1 + m^2)^s |V_{2m}|^2 < \infty,$$

for some $s \geq 0$, where $V_0 = 0$ and $V_{2m} = V_{-2m} = \bar{V}_{-2m}$.

Definition: The moving gap soliton of the coupled-mode system is said to be a reversible homoclinic orbit if (A, B) decays to zero at infinity and $A(\xi) = \bar{A}(-\xi)$, $B(\xi) = \bar{B}(-\xi)$ in the parametrization above.

Remark: If $V(x) = V(-x)$ and $U(x)$ is a solution of $\nabla^2 U + \omega U = V(x) U + \sigma |U|^2 U$, then $\bar{U}(-x)$ is also a solution.

Main Theorem

Theorem: Let $V(x)$ satisfy the assumption and $V_{2n} \neq 0$ for a $n \in \mathbb{N}$.
Let $\omega = \frac{n^2}{4} + \epsilon\Omega$ with $|\Omega| < \Omega_0 = |V_{2n}|\frac{\sqrt{n^2-c^2}}{n}$.
Let $0 < c < n$, such that $\frac{n^2+c^2}{2c} \notin \mathbb{Z}$. Fix $N \in \mathbb{N}$.
Then, there exists $\epsilon_0, L, C > 0$ such that for all $\epsilon \in (0, \epsilon_0)$ the Gross–Pitaevskii equation has a solution in the form $E(x,t) = e^{-i\omega t}\psi(x,y)$, where $y = x - ct$ and the function $\psi(x,y)$ is a periodic (anti-periodic) function of x for even (odd) n, satisfying the reversibility constraint $\psi(x,y) = \bar{\psi}(x,-y)$, and

$$\left|\psi(x,y) - \epsilon^{1/2}\left(a_\epsilon(\epsilon y)e^{\frac{inx}{2}} + b_\epsilon(\epsilon y)e^{-\frac{inx}{2}}\right)\right| \le C_0\epsilon^{N+1/2},$$

for all $x \in \mathbb{R}$ and $y \in [-L/\epsilon^{N+1}, L/\epsilon^{N+1}]$. Here $a_\epsilon(Y) = a(Y) + \mathrm{O}(\epsilon)$ on $Y = \epsilon y \in \mathbb{R}$ is an exponentially decaying reversible solution, while $a(Y)$ is a solution of the coupled-mode system with $Y = X - cT$.

Remarks on the Main Theorem

1. The solution $\psi(x, y)$ is a bounded non-decaying function on a large finite interval

$$y \in [-L/\epsilon^{N+1}, L/\epsilon^{N+1}] \subset \mathbb{R}$$

   and we do not claim that the solution $\psi(x, y)$ can be extended to a global bounded function on $y \in \mathbb{R}$.

2. Since the homoclinic orbit (a, b) of the coupled-mode system is single-humped, the traveling solution $\psi(x, y)$ is represented by a single bump surrounded by bounded oscillatory tails.

3. The solution (a_ϵ, b_ϵ) is defined up to the terms of $\mathrm{O}(\epsilon^N)$ and it satisfies an extended coupled-mode system, which is a perturbation of the coupled-mode system with $Y = X - cT$.

Spatial dynamics formulation

Set $E(x,t) = e^{-i\omega t}\psi(x,y)$ with $y = x - ct$ and a parameter ω. For traveling solutions, $c \neq 0$ and we set $c > 0$. Then,

$$\left(\omega - ic\partial_y + \partial_x^2 + 2\partial_x\partial_y + \partial_y^2\right)\psi = V(x)\psi + \sigma|\psi|^2\psi.$$

We consider functions $\psi(x,y)$ being 2π-periodic or 2π-antiperiodic in x and bounded in y. Therefore,

$$\psi(x,y) = \sum_{m \in \mathbb{Z}'} \psi_m(y) e^{\frac{i}{2}mx},$$

such that $\psi_m(y)$ satisfy the nonlinear system of coupled ODEs:

$$\psi_m'' + i(m-c)\psi_m' + \left(\omega - \frac{m^2}{4}\right)\psi_m = \sum_{m_1 \in \mathbb{Z}'} V_{m-m_1}\psi_{m_1} + \mathrm{N.T.}$$

Eigenvalues of the spatial dynamics

Linearization of the system with $\psi_m(y) = e^{\kappa y}\delta_{m,m_0}$ gives roots $\kappa = \kappa_m$ in the quadratic equation with $\omega = \frac{n^2}{4}$:

$$\kappa^2 + i(m-c)\kappa + \omega - \frac{m^2}{4} = 0, \qquad \forall m \in \mathbb{Z}'.$$

- For $m > m_0 = \left[\frac{n^2+c^2}{2c}\right]$, all roots are complex-valued.
- For $m \leq m_0$, all roots are purely imaginary. The zero root is semi-simple of multiplicity two. All other roots are semi-simple of maximal multiplicity three.
- If c is irrational, all non-zero roots are simple but may approach to each other arbitrarily closer.

Hamiltonian formulation

Let $\phi_m(y) = \psi'_m(y) - \frac{i}{2}(c-m)\psi_m(y)$ and rewrite the system of ODEs:

$$\begin{cases} \frac{d\psi_m}{dy} & = & \phi_m + \frac{i}{2}(c-m)\psi_m \\ \frac{d\phi_m}{dy} & = & -\frac{1}{4}\left(n^2+c^2-2cm\right)\psi_m + \frac{i}{2}(c-m)\phi_m - \epsilon\Omega\psi_m + \mathrm{N.T.} \end{cases}$$

The system is Hamiltonian in canonical variables $(\boldsymbol{\psi}, \boldsymbol{\phi}, \bar{\boldsymbol{\psi}}, \bar{\boldsymbol{\phi}})$. The vector field maps a domain in D to a range in X, where

$$D = \left\{(\boldsymbol{\psi}, \boldsymbol{\phi}, \bar{\boldsymbol{\psi}}, \bar{\boldsymbol{\phi}}) \in l^2_{s+1}(\mathbb{Z}, \mathbb{C}^4)\right\}, \;\; X = \left\{(\boldsymbol{\psi}, \boldsymbol{\phi}, \bar{\boldsymbol{\psi}}, \bar{\boldsymbol{\phi}}) \in l^2_{s}(\mathbb{Z}, \mathbb{C}^4)\right\}$$

and $l^2_s(\mathbb{Z})$ is a Banach algebra for any $s > \frac{1}{2}$. The phase space is X.

Symmetries

Solutions are invariant under the reversibility transformation

$$\boldsymbol{\psi}(y) \mapsto \bar{\boldsymbol{\psi}}(-y), \quad \boldsymbol{\phi}(y) \mapsto -\bar{\boldsymbol{\phi}}(-y), \quad \forall y \in \mathbb{R}.$$

and the gauge transformation

$$\boldsymbol{\psi}(y) \mapsto e^{i\alpha}\boldsymbol{\psi}(y), \;\; \boldsymbol{\phi}(y) \mapsto e^{i\alpha}\boldsymbol{\phi}(y), \qquad \forall \alpha \in \mathbb{R}.$$

Reversible solutions satisfy the constraints:

$$\boldsymbol{\psi}(-y) = \bar{\boldsymbol{\psi}}(y), \;\; \boldsymbol{\phi}(-y) = -\bar{\boldsymbol{\phi}}(y), \;\; \forall y \in \mathbb{R},$$

which means that the trajectory intersects the reversibility surface

$$\Sigma_r = \left\{ (\boldsymbol{\psi}, \boldsymbol{\phi}, \bar{\boldsymbol{\psi}}, \bar{\boldsymbol{\phi}}) \in D : \quad \mathrm{Im}\boldsymbol{\psi} = 0, \;\; \mathrm{Re}\boldsymbol{\phi} = 0 \right\}.$$

Canonical transformations

Let $\mathbb{Z}_- = \{m \in \mathbb{Z}' : \ m \le m_0\}$, $\mathbb{Z}_+ = \{m \in \mathbb{Z}' : \ m > m_0\}$ and

$$\mathbb{Z}_- : \psi_m = \frac{c_m^+ + c_m^-}{\sqrt[4]{n^2 + c^2 - 2cm}}, \phi_m = \frac{i}{2}\sqrt[4]{n^2 + c^2 - 2cm}(c_m^+ - c_m^-),$$

$$\mathbb{Z}_+ : \psi_m = \frac{c_m^+ + c_m^-}{\sqrt[4]{2cm - n^2 - c^2}}, \phi_m = \frac{1}{2}\sqrt[4]{2cm - n^2 - c^2}(c_m^+ - c_m^-).$$

The new Hamiltonian system is rewritten in new canonical variables

$$\forall m \in \mathbb{Z}_- : \qquad \frac{dc_m^+}{dy} = i\frac{\partial H}{\partial \bar{c}_m^+}, \quad \frac{dc_m^-}{dy} = -i\frac{\partial H}{\partial \bar{c}_m^-},$$

$$\forall m \in \mathbb{Z}_+ : \qquad \frac{dc_m^+}{dy} = -\frac{\partial H}{\partial \bar{c}_m^-}, \quad \frac{dc_m^-}{dy} = \frac{\partial H}{\partial \bar{c}_m^+},$$

where H is a new Hamiltonian functions in variables $\mathbf{c}^+$ and $\mathbf{c}^-$.

Truncated coupled-mode system

The new Hamiltonian function is

$$H = \sum_{m \in \mathbb{Z}_-} \left(k_m^+ |c_m^+|^2 - k_m^- |c_m^-|^2 \right) + \sum_{m \in \mathbb{Z}_+} \left(\kappa_m^- c_m^- \bar{c}_m^+ - \kappa_m^+ c_m^+ \bar{c}_m^- \right) + \mathrm{N.T.}$$

Consider the subspace

$$S = \left\{ c_m^+ = 0, \; \forall m \in \mathbb{Z} \backslash \{n\}, \quad c_m^- = 0, \; \forall m \in \mathbb{Z} \backslash \{-n\} \right\}$$

and truncate H on the subspace S:

$$H|_S = \epsilon \left[\frac{\Omega |c_n^+|^2}{n - c} + \frac{\Omega |c_{-n}^-|^2}{n + c} - \frac{V_{2n} (\bar{c}_n^+ c_{-n}^- + c_n^+ \bar{c}_{-n}^-)}{\sqrt{n^2 - c^2}} + \mathrm{N.T.} \right].$$

The Hamiltonian system for (c_n^+, c_n^-) is nothing but the coupled-mode system for $a = \frac{c_n^+}{\sqrt{n-c}}$ and $b = \frac{c_{-n}^-}{\sqrt{n+c}}$ in $Y = \epsilon y$.

Extended coupled-mode system

How to avoid formal truncation and to separate the coupled-mode system from the remainder? Use near-identity canonical transformations to obtain the new Hamiltonian function in the form

$$H = \sum_{m \in \mathbb{Z}_-} \left(k_m^+ |c_m^+|^2 - k_m^- |c_m^-|^2 \right) + \sum_{m \in \mathbb{Z}_+} \left(\kappa_m^- c_m^- \bar{c}_m^+ - \kappa_m^+ c_m^+ \bar{c}_m^- \right)$$

$$+\epsilon H_S(c_n^+, c_{-n}^-) + \epsilon H_T(c_n^+, c_{-n}^-, \mathbf{c}^+, \mathbf{c}^-) + \epsilon^{N+1} H_R(c_n^+, c_{-n}^-, \mathbf{c}^+, \mathbf{c}^-).$$

If $H_R \equiv 0$, the subspace S is invariant subspace of the Hamiltonian system and dynamics on S is given by a four-dimensional ODE system

$$\frac{dc_n^+}{dY} = i \frac{\partial H_S}{\partial \bar{c}_n^+}, \qquad \frac{dc_{-n}^-}{dY} = -i \frac{\partial H_S}{\partial \bar{c}_{-n}^+},$$

where $Y = \epsilon y$.

Persistence results

Lemma: There exists a reversible homoclinic orbit of the extended coupled-mode system which satisfies

$$|c_n^+(y)| \leq C_+ e^{-\epsilon\gamma|y|}, \quad |c_{-n}^-(y)| \leq C_- e^{-\epsilon\gamma|y|}, \quad \forall y \in \mathbb{R},$$

for some $\gamma, C_+, C_- > 0$ and sufficiently small ϵ.

Lemma: The linearized system at the zero solution is topologically equivalent for sufficiently small ϵ, except that the double zero eigenvalue at $\epsilon = 0$ split into a pair of complex eigenvalues to the left and right half-planes for $\epsilon > 0$.

Divide the phase space near the zero solution into

$$X = X_h \oplus X_c \oplus X_u \oplus X_s$$

and rewrite the system for $\mathbf{c}_0 + \mathbf{c}_h \in X_h$ and $\mathbf{c} \in X_c \oplus X_u \oplus X_s$.

Final system of equations

The system of equations

$$\begin{aligned}
\frac{d\mathbf{c}_h}{dy} &= \epsilon\Lambda_h(\mathbf{c}_0)\mathbf{c}_h + \epsilon\mathbf{G}_T(\mathbf{c}_0)(\mathbf{c}_h,\mathbf{c}) + \epsilon^{N+1}\mathbf{G}_R(\mathbf{c}_0+\mathbf{c}_h,\mathbf{c}),\\
\frac{d\mathbf{c}}{dy} &= \Lambda_\epsilon\mathbf{c} + \epsilon\mathbf{F}_T(\mathbf{c}_0+\mathbf{c}_h,\mathbf{c}) + \epsilon^{N+1}\mathbf{F}_R(\mathbf{c}_0+\mathbf{c}_h,\mathbf{c}),
\end{aligned}$$

where the linearization operator $\Lambda_h(\mathbf{c}_0)$ has a two-dimensional kernel spanned by $\mathbf{c}_0'(y)$ and $\sigma_1\mathbf{c}_0(y)$ and the remainder terms satisfy the bounds

$$\begin{aligned}
\|\mathbf{G}_R\|_{X_h} &\leq N_R\left(\|\mathbf{c}_0+\mathbf{c}_h\|_{X_h} + \|\mathbf{c}\|_{X_h^\perp}\right),\\
\|\mathbf{G}_T\|_{X_h} &\leq N_T\left(\|\mathbf{c}_h\|_{X_h}^2 + \|\mathbf{c}\|_{X_h^\perp}^2\right),\\
\|\mathbf{F}_T\|_{X'} &\leq M_T\left(\|\mathbf{c}_0+\mathbf{c}_h\|_{X_h} + \|\mathbf{c}\|_{X_h^\perp}\right)\|\mathbf{c}\|_{X_h^\perp}.
\end{aligned}$$

Local center-stable manifold

Theorem: Let $\mathbf{a} \in X_c$, $\mathbf{b} \in X_s$ and $(\alpha_1, \alpha_2) \in \mathbb{C}^2$ be small:

$$\|\mathbf{a}\|_{X_c} \leq C_a \epsilon^N, \quad \|\mathbf{b}\|_{X_s} \leq C_b \epsilon^N, \quad |\alpha_1| + |\alpha_2| \leq C_\alpha \epsilon^N.$$

There exists a family of local solutions $\mathbf{c}_h = \mathbf{c}_h(y; \mathbf{a}, \mathbf{b}, \alpha_1, \alpha_2)$ and $\mathbf{c} = \mathbf{c}(y; \mathbf{a}, \mathbf{b}, \alpha_1, \alpha_2)$ such that

$$\mathbf{c}_c(0) = \mathbf{a}, \quad \mathbf{c}_s = e^{y\Lambda_s}\mathbf{b} + \tilde{\mathbf{c}}_s(y), \quad \mathbf{c}_h = \alpha_1 \mathbf{s}_1(y) + \alpha_2 \mathbf{s}_2(y) + \tilde{\mathbf{c}}_h(y),$$

where $\tilde{\mathbf{c}}_s(y)$ and $\tilde{\mathbf{c}}_h(y)$ are uniquely defined and the family of local solutions satisfies the bound

$$\sup_{\forall y \in [0, L/\epsilon^{N+1}]} \|\mathbf{c}_h(y)\|_{X_h} \leq C_h \epsilon^N, \qquad \sup_{\forall y \in [0, L/\epsilon^{N+1}]} \|\mathbf{c}(y)\|_{X_h^\perp} \leq C \epsilon^N,$$

for some constants $C_h, C > 0$.

Ideas of the proof

1. Use the cut-off function on $y \in [0, y_0]$ and use the Implicit Function Theorem for components $\mathbf{c}_s, \mathbf{c}_u$ resulting in

$$\|\mathbf{c}_{s,u}\|_{C_b^0} \leq C\|\mathbf{F}_{s,u}\|_{C_b^0}.$$

2. Use the cut-off functions on $y \in [0, y_0]$ and the reversible continuation of solutions on $y \in [-y_0, 0]$. Then, use the Implicit Function Theorem for component $\mathbf{c}_h$ resulting in

$$\|\mathbf{c}_h\|_{C_b^0} \leq \frac{C}{\epsilon}\|\mathbf{F}_h\|_{C_b^0}.$$

3. Use variation of constant formula and the Gronwall inequality for component $\mathbf{c}_c$. The bounds are consistent for $y_0 = L/\epsilon^{N+1}$.

Proof of the main theorem

The local center-stable manifold is extended to a local solution on $y \in [-y_0, y_0]$ if it intersects the reversibility surface Σ_r.

Since $\mathbf{c}_c(0) = \mathbf{a}$ is arbitrary, we can set immediately

$$\mathrm{Im}(\mathbf{a})^+_m = 0, \ \forall m \in \mathbb{Z}_- \backslash \{n\}, \ \ \mathrm{Im}(\mathbf{a})^-_m = 0, \ \forall m \in \mathbb{Z}_- \backslash \{-n\}.$$

The other parameters $\mathbf{b}$ and (α_1, α_2) are not however the initial conditions. They satisfy the set of reversibility constraints

$$\mathrm{Re} b_m + \mathrm{Re}(\tilde{\mathbf{c}}_s)_m(0) = \mathrm{Re}(\mathbf{c}_u)_m(0), \ \mathrm{Im} b_m + \mathrm{Im}(\tilde{\mathbf{c}}_s)_m(0) = -\mathrm{Im}(\mathbf{c}_u)_m(0$$

and

$$\mathrm{Im} c^+_n(0) = 0, \quad \mathrm{Im} c^-_{-n}(0) = 0.$$

The first set is solved by the Implicit Function Theorem. The second set is satisfied if $\alpha_1 = \alpha_2 = 0$, since the kernel does not satisfy the reversibility but the inhomogeneous solution for $\mathbf{c}_h$ does.

Extensions

We have checked that modified Gross–Pitaevskii equations still possess infinitely many eigenvalues on the imaginary axis:

$$
\begin{aligned}
E_{tt} &= E_{xx} + V(x)E + \sigma|E|^2 E, \\
iE_t &= -E_{xx} + iE_{xxt} + V(x)E + \sigma|E|^2 E, \\
i\dot{E}_n &= -E_{n+1} - E_{n-1} + V_n E_n + \sigma|E_n|^2 E_n.
\end{aligned}
$$

In all these equations, there is no hope to construct true homoclinic solution (moving gap soliton) but one can construct a local reversible center-stable manifold, which resembles a single bump surrounded by oscillatory tails.

It is an open problem how to extend this local solution to a global solution defined on the entire line.