Solitary waves under intensity-dependent dispersion

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joint work with Panos Kevrekidis and Ryan Ross
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and with Michael Plum (Karlsruhe Institute of Technology)

Dedicated to Dimitri's Frantzeskakis 60th birthday

Bright soliton
$\psi(t,x) = e^{it}\text{sech}(x)$
of the focusing NLS equation

$$i\partial_t \psi + \partial_x^2 \psi + 2|\psi|^2\psi = 0$$

satisfying $|\psi(t,x)| \to 0$ as $|x| \to \infty$

Dark soliton
$\psi(t,x) = e^{-2it}\tanh(x)$
of the defocusing NLS equation

$$i\partial_t \psi + \partial_x^2 \psi - 2|\psi|^2\psi = 0$$

satisfying $|\psi(t,x)| \to 1$ as $|x| \to \infty$

NLS with intensity-dependent dispersion

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Taking into account higher-order nonlinearity and dispersion gives an extended version of the NLS equation:

$$i\psi_t + \psi_{xx} + |\psi|^2\psi + ic_1\psi_{xxx} + ic_2|\psi|^2\psi_x + ic_3(|\psi|^2\psi)_x + c_4|\psi|^4\psi = 0.$$

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What we study is a different model where the dispersion coefficient depends on the wave intensity:

$$i\psi_t + (1 - |\psi|^2)\psi_{xx} = 0. \tag{NLS-IDD}$$

C.Y. Lin, J.H. Chang, G. Kurizki, and R.K. Lee, Optics Letters **45** (2020), 1471–1474

NLS with intensity-dependent dispersion

Two conserved quantities exist for NLS-IDD:

$$Q(\psi) = -\int_{\mathbb{R}} \log|1 - |\psi|^2| dx$$

and

$$E(\psi) = \int_{\mathbb{R}} |\psi_x|^2 dx.$$

Standing waves exist in the form $\psi(x,t) = e^{i\omega t}u(x)$ with (ω, u) satisfying

$$\omega u(x) = (1 - u^2)u''(x).$$

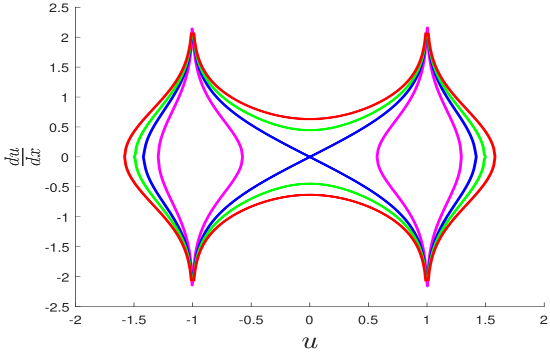
Solitary waves with $u(x) \to 0$ as $|x| \to \infty$ exist only if $\omega > 0$, in which case ω can be scaled out by $u(x) \mapsto u(\sqrt{\omega}x)$.

Phase plane portrait

Equation $(1-u^2)u'' = u$ is integrable with the first invariant:

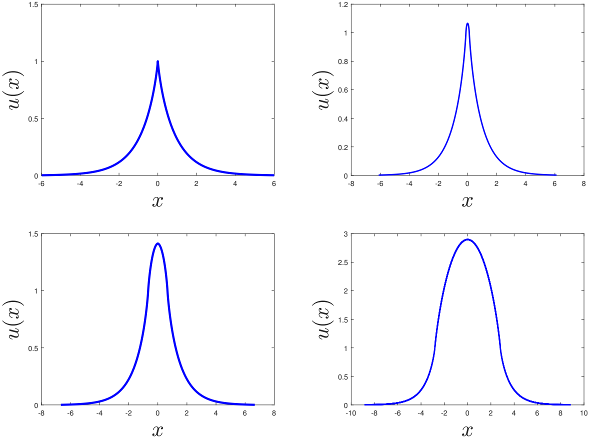
$$\frac{1}{2}\left(\frac{du}{dx}\right)^2 + \frac{1}{2}\log|1-u^2| = C,$$

where C is constant. The solution is singular at $u = \pm 1$.



Possible solitary waves

Gluing the stable and unstable curves with another integral curves give a one-parameter family of single-humped solitary waves:



Top left: “cusped soliton". Bottom left: “bell-shaped soliton".

Questions on existence and stability of these solitary waves

- In what space (in what sense) do they exist?
- What is the nature of singularity at $u = \pm 1$?
- Can these solutions be characterized variationally?
- Are they stable in the time evolution of the NLS-IDD?

Existence result

Definition

We say that $u \in H^1(\mathbb{R})$ is a weak solution of the differential equation $u = (1-u^2)u''$ if it satisfies the following equation

$$\langle u, \varphi \rangle + \langle (1-u^2)u', \varphi' \rangle - 2\langle u(u')^2, \varphi \rangle = 0, \quad \text{for every } \varphi \in H^1(\mathbb{R}),$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(\mathbb{R})$.

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where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(\mathbb{R})$.

Theorem (Ross–Kevrekidis–P, Q.Appl.Math. 79 (2021) 641)

There exists a one-parameter continuous family of weak, positive, and single-humped solutions of $u = (1 - u^2)u''$ parametrized by C.

What is needed for the proof beyond the phase plane analysis:

- ▷ $u \in H^1(\mathbb{R})$;
- ▷ $\lim\limits_{x \to x_0} (1 - u^2(x))u'(x) = 0$ for each x_0 where $u(x_0) = 1$.

Nature of singularity at $u = 1$

It follows from the first invariant

$$\frac{1}{2}\left(\frac{du}{dx}\right)^2 + \frac{1}{2}\log|1-u^2| = C,$$

that the cusped soliton is defined by the implicit function

$$|x| = \int_u^1 \frac{d\xi}{\sqrt{-\log(1-\xi^2)}}, \quad u \in (0,1).$$

Asymptotic analysis gives as $|x| \to 0$:

$$u(x) = 1 - |x|\sqrt{\log(1/|x|)}\left[1 + \mathcal{O}\left(\frac{\log\log(1/|x|)}{\log(1/|x|)}\right)\right].$$

[Alfimov–Korobeinikov–Lustri–P, Nonlinearity 32 (2019) 3445]

Hence, $u'(x) \sim \sqrt{\log(1/|x|)}$ and $(1-u^2)u'(x) \sim |x|\log(1/|x|)$.

Numerical illustration of the asymptotic profile

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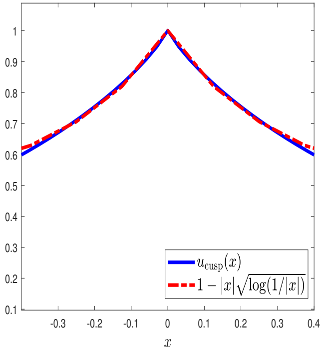

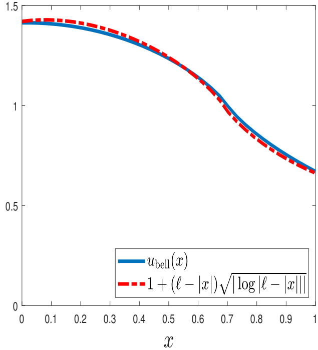

Left: “cusped soliton". Right: “bell-shaped soliton".

Towards the stability result

Recall the conserved quantities:

$$Q(\psi) = -\int_{\mathbb{R}} \log|1 - |\psi|^2| dx, \quad E(\psi) = \int_{\mathbb{R}} |\psi_x|^2 dx.$$

Solitary wave $\psi(x,t) = u(x)e^{i\omega t}$ is a critical point of the action

$$\Lambda_\omega(u) = E(u) + \omega Q(u),$$

however, the formal expansion yields

$$\Lambda_\omega(u+\varphi) - \Lambda_\omega(u) = 2\langle u', \varphi' \rangle + 2\langle (1-u^2)^{-1} u, \varphi \rangle + \mathcal{O}(\|\varphi'\|_{L^2}^2 + \|(1-u^2)^{-1}\varphi\|_{L^2 \cap L^\infty}^2),$$

which is not compatible with the definition of weak solutions:

$$u \in H^1(\mathbb{R}) : \quad \omega\langle u, \varphi \rangle + \langle (1-u^2)u', \varphi' \rangle - 2\langle u(u')^2, \varphi \rangle = 0,$$

for every $\varphi \in H^1(\mathbb{R})$.

New definition of weak solutions

Definition

Fix $L > 0$ and define

$$X_L := \left\{ u \in H^1(\mathbb{R}) : \ u(x) > 1, \ x \in (-L, L) \text{ and } u(x) \leq 1, \ |x| \geq L \right\}.$$

Pick $u_L \in X_L$ satisfying

$$\lim_{|x| \to L} \frac{u_L(x) - 1}{(L - |x|)\sqrt{|\log|L - |x|||}} = 1.$$

We say that $u \in X_L \subset H^1(\mathbb{R})$ is a weak solution if it satisfies the following equation

$$\langle u', \varphi' \rangle + \omega \langle (1 - u^2)^{-1} u, \varphi \rangle = 0, \quad \text{for every } \varphi \in H^1_L,$$

where $H^1_L := \left\{ \varphi \in H^1(\mathbb{R}) : \ \ (1 - u_L^2)^{-1} \varphi \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) \right\}$.

Stability result

Theorem (P–Ross–Kevrekidis, J. Phys. A 54 (2021) 445701)

For every $\mu > 0$ and $L > 0$, there exists a minimizer of the constrained variational problem

$$\mathcal{Q}_{\mu,L} := \inf_{u \in X_L} \{ Q(u) : \quad E(u) = \mu \}.$$

The minimizer coincides with a scaled version of the one-parameter family of solitary waves for $C = C_{\mu L}$.

What is needed for the proof beyond the expansion of Λ_ω in X_L:

- Monotonicity of mappings $C \mapsto E(u_C)$ and $C \mapsto \ell_C$, where $2\ell_C$ is the length of the bell head;
- Scaling transformation;
- Convexity of action Λ_ω at u_C.

Monotonicity of mappings $C \mapsto E(u_C)$ and $C \mapsto \ell_C$

It follows from $(u')^2 + \log|1-u^2| = 2C$ that

$$E(u_C) = E(u_{\text{cusp}}) + 2\int_1^{\sqrt{1+e^{2C}}} \sqrt{2C - \log(u^2-1)}du$$

and

$$\ell_C = \int_1^{\sqrt{1+e^{2C}}} \frac{du}{\sqrt{2C - \log(u^2-1)}}$$

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$\frac{dE(u_C)}{dC} > 0$ follows from

$$\frac{dE(u_C)}{dC} = 2\int_1^{\sqrt{1+e^{2C}}} \frac{du}{\sqrt{2C - \log(u^2 - 1)}} = 2\ell_C.$$

Monotonicity of mappings $C \mapsto E(u_C)$ and $C \mapsto \ell_C$

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$\frac{d\ell_C}{dC} > 0$ follows from a longer computation, where we use **the period function** for periodic orbits on the phase plane.

Monotonicity of mappings $C \mapsto E(u_C)$ and $C \mapsto \ell_C$

It follows from $(u')^2 + \log|1-u^2| = 2C$ that

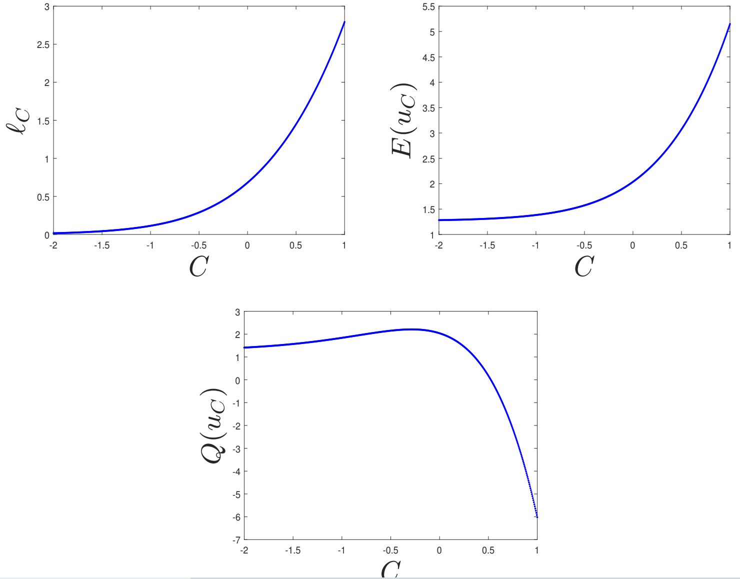
$$E(u_C) = E(u_{\rm cusp}) + 2\int_1^{\sqrt{1+e^{2C}}} \sqrt{2C - \log(u^2-1)}du$$

and

$$\ell_C = \int_1^{\sqrt{1+e^{2C}}} \frac{du}{\sqrt{2C - \log(u^2-1)}}$$

The mapping $C \mapsto Q(u_C)$ is non-monotone.

Numerical illustrations



Scaling transformation

Recall the variational problem for $\mu > 0$ and $L > 0$:

$$\mathcal{Q}_{\mu,L} := \inf_{u \in X_L} \{Q(u) : \quad E(u) = \mu\},$$

with the Euler–Lagrange equation $\omega u = (1 - u^2)u''$.

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with the Euler–Lagrange equation $\omega u = (1 - u^2)u''$.

Let u_C be a solution of $u = (1 - u^2)u''$. Then, $u_\omega(x) = u_C(\sqrt{\omega}x)$ is a solution of the Euler–Lagrange equation so that

$$Q(u_\omega) = \frac{1}{\sqrt{\omega}} Q(u_C), \quad E(u_\omega) = \sqrt{\omega} E(u_C)$$

and

$$L = \frac{1}{\sqrt{\omega}} \ell_C, \quad \mu = \sqrt{\omega} E(u_C).$$

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with the Euler–Lagrange equation $\omega u = (1 - u^2)u''$.

Transformation $(\omega, C) \mapsto (\mu, L)$ is invertible because the Jacobian is

$$\begin{vmatrix} \frac{\partial \mu}{\partial \omega} & \frac{\partial \mu}{\partial C} \\ \frac{\partial L}{\partial \omega} & \frac{\partial L}{\partial C} \end{vmatrix} = \frac{1}{2\omega} \left[E(u_C) \frac{d\ell_C}{dC} + \ell_C \frac{dE(u_C)}{dC} \right] > 0.$$

Hence the mapping $(\omega, C) \mapsto (\mu, L)$ is invertible and there exists a unique $C = C_{\mu,L}$ for every $\mu > 0$ and $L > 0$. In fact, $\ell_C E(u_C) = L\mu$.

Convexity of action Λ_ω

Let $v + iw$ with real $v, w \in H^1_{\ell_C} \subset H^1(\mathbb{R})$ be a perturbation to u_C. Then, the action is expanded as

$$\Lambda_{\omega=1}(u_C + v + iw) = \Lambda_{\omega=1}(u_C) + Q_+(v) + Q_-(w) + R(v,w),$$

where $R(v,w)$ is the remainder term

$$R(v,w) = \int_{\mathbb{R}} \left[\log\left(1 - \frac{2u_C v + v^2 + w^2}{1-u_C^2}\right) + \frac{2u_C v}{1-u_C^2} + \frac{(1+u_C^2)v^2}{(1-u_C^2)^2} + \frac{w^2}{1-u_C^2} \right] dx.$$

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$R(v,w)$ is cubic with respect to perturbation:

$$|R(v,w)| \leq C\|(1-u_C^2)^{-1}v\|^3_{L^2\cap L^\infty} + C\|(1-u_C^2)^{-1}w\|^3_{L^2\cap L^\infty},$$

Convexity of action Λ_ω

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Then, the action is expanded as

$$\Lambda_{\omega=1}(u_C + v + iw) = \Lambda_{\omega=1}(u_C) + Q_+(v) + Q_-(w) + R(v, w),$$

whereas Q_+ and Q_- are the quadratic forms:

$$Q_+(v) = \int_{\mathbb{R}} \left[(v_x)^2 + \frac{(1 + u_C^2)v^2}{(1 - u_C^2)^2} \right] dx, Q_-(w) = \int_{\mathbb{R}} \left[(w_x)^2 + \frac{w^2}{1 - u_C^2} \right] dx,$$

For cusped soliton with $0 < u \leq 1$, they are coercive and bounded as

$$Q_\pm(v) \geq \|v\|^2_{H^1}, \quad Q_\pm(v) \leq C_\pm \left(\|v'\|^2_{L^2} + \|(1 - u_C^2)^{-1} v\|^2_{L^2} \right)$$

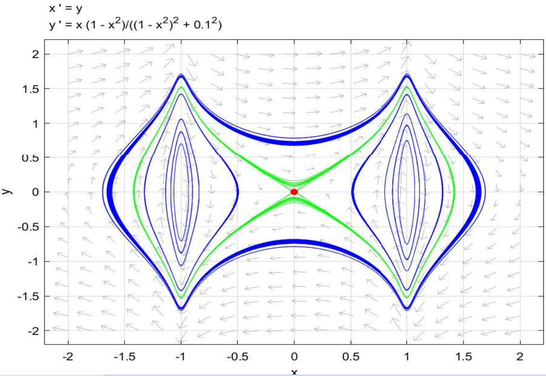
Hence $u_{C_{\mu L}}$ is a minimizer of $Q(u)$ in X_L for fixed $L > 0$ and $\mu > 0$.

Construction of solitons with numerical methods

Regularization: Replace $u = (1-u^2)u''$ by

$$u_\epsilon'' = \frac{u_\epsilon(1-u_\epsilon^2)}{(1-u_\epsilon^2)^2+\epsilon^2},$$

for fixed $\epsilon > 0$. Only bell-shaped soliton is found.



Construction of solitons with numerical methods

Petviashvili's method: Rewrite $u = (1 - u^2)u''$ as $u - u'' = -u^2 u''$ and interpret a solution $u \in H^1(\mathbb{R})$ as a fixed point $u = T(u)$ of the nonlinear operator

$$T(u) := -(1 - \partial_x^2)^{-1} u^2 \partial_x^2 u.$$

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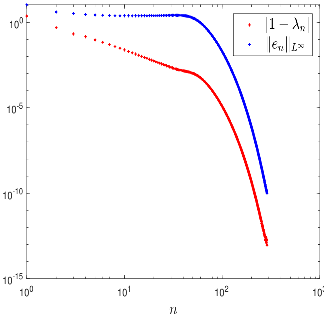
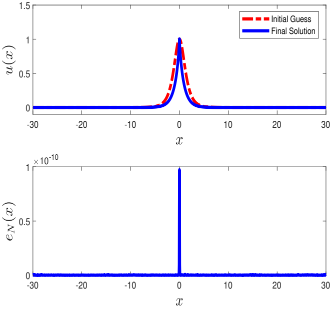




Construction of solitons with numerical methods

Newton's method: Represent a solution of $u = (1-u^2)u''$ as a root of the nonlinear equation $F(u) = 0$, where

$$F(u) := -(1-u^2)\partial_x^2 u + u.$$

Roots of the nonlinear equation $F(u) = 0$ in $H^1(\mathbb{R})$ can be approximated by using the Newton iterations:

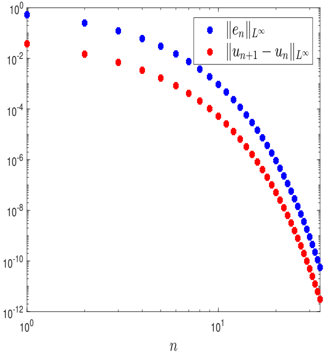
$$u_{n+1} = u_n - \mathcal{L}^{-1} F(u_n),$$

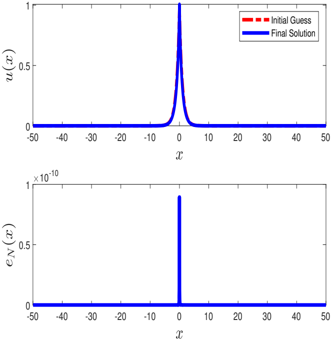
where $\mathcal{L} := -(1-u^2)\partial_x^2 + \frac{1+u^2}{1-u^2}$ is invertible.

All solitary waves of the family u_C can be recovered by using this numerical method.

Construction of solitons with numerical methods

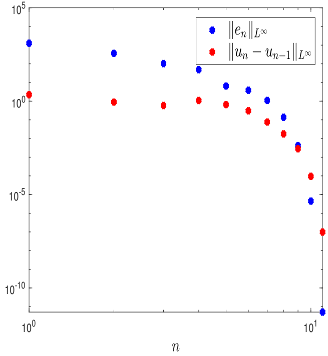
Here iterations approach the cusped soliton.

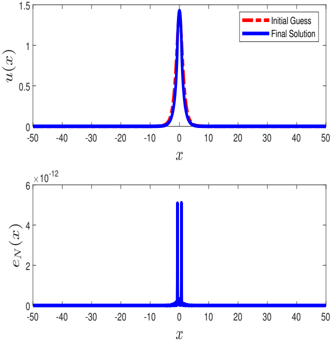




Construction of solitons with numerical methods

Here iterations approach the bell-shaped soliton.





Summary on bright solitons

We considered NLS equation with intensity-dependent dispersion

$$i\psi_t + (1 - |\psi|^2)\psi_{xx} = 0.$$

- Continuum of singular solitary waves exists $\psi(x,t) = u_C(x)e^{it}$.
- Each solitary wave can be characterized as a minimizer of mass for fixed energy and fixed distance between two singularities.
- Solitary waves can be approximated numerically both in the stationary and time-dependent problems.
- Well-posedness of the model is opened for further studies.

Further development: dark solitons

Together with M. Plum (KIT), we were looking for NLS models for which the spectrum of linearized operator at the dark soliton consists of isolated eigenvalues. This is true for the NLS equation with the inverse intensity-dependent dispersion

$$i(1 - |\psi|^2)\psi_t + \psi_{xx} = 0.$$

With the transformation $\psi(x,t) = u(x,t)e^{2it}$, the model becomes similar to the cubic defocusing NLS equation

$$i(1 - |u|^2)u_t + u_{xx} + 2(1 - |u|^2)u = 0,$$

which admit the black soliton in the form $u(x) = \tanh(x)$.

The dark solitons $u(t,x) = U_c(x - 2ct)$ are found from

$$U_c'' - 2ic(1 - |U_c|^2)U_c' + 2(1 - |U_c|^2)U_c = 0.$$

Time evolution

Solutions can be considered in the set $\mathcal{F}$,

$$\mathcal{F} := \{ f \in L^\infty(\mathbb{R}) : \quad |f(x)| < 1, \;\; x \in \mathbb{R}, \;\; |f(x)| \to 1 \;\; \text{as} \;\; |x| \to \infty \},$$

althought it is not known if the set $\mathcal{F}$ is invariant under the time evolution. Dark solitons exist in $U_c \in \mathcal{F}$ for every $c \in \mathbb{R}$.

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Conserved quantities of mass and energy

$$M(\psi) = \int (1 - |\psi|^2)^2 dx, \quad E(\psi) = \int |\psi_x|^2 dx$$

and the momentum

$$P(\psi) = \frac{1}{2i} \int \frac{(1 - |\psi|^2)^2}{|\psi|^2} (\bar{\psi}\psi_x - \bar{\psi}_x \psi) dx.$$

Conservation is proven for $\psi(t,x) = e^{i\theta_\pm}(1 + \mathcal{O}(e^{-\alpha_\pm |x|})), x \to \pm\infty$.

Main result 1: linearization at the black soliton

Using the decomposition $\psi(t,x) = e^{-2it}[\varphi(x) + u(t,x) + iv(t,x)]$, where $\varphi(x) = \tanh(x)$ and $u + iv$ is the perturbation, we obtain the linearized equations of motion

$$(1-\varphi^2)u_t = L_- v, \quad (1-\varphi^2)v_t = -L_+ u,$$

where $L_+ = -\partial_x^2 + 4 - 6\mathrm{sech}^2(x)$ and $L_- = -\partial_x^2 - 2\mathrm{sech}^2(x)$ are the same as in the NLS equation.

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The spectral problem

$$L_- v = \lambda(1-\varphi^2)u, \quad L_+ u = -\lambda(1-\varphi^2)v$$

is defined in the Hilbert space $\mathcal{H}$ with the inner product

$$(f,g)_{\mathcal{H}} := \int (1-\varphi^2)\bar{f} g dx = \int \text{sech}^2(x)\bar{f}(x)g(x)dx.$$

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Theorem

- *The spectrum of L_+ in $\mathcal{H}$ consists of simple eigenvalues $\mu_n = n(n+5)$, $n \geq 0$.*
- *The spectrum of L_- in $\mathcal{H}$ consists of simple eigenvalues $\nu_n = n(n+1) - 2$, $n \geq 0$.*
- *The spectrum of the stability problem in $\mathcal{H} \times \mathcal{H}$ consists of pairs of isolated eigenvalues $\{\pm i\omega_1, \pm i\omega_2, \cdots\}$ and zero eigenvalue.*

Main result 2: stability of the black soliton

Expanding the energy functional

$$\Lambda(\psi) := \int [|\psi_x|^2 + (1 - |\psi|^2)^2] dx$$

at the black soliton $\varphi(x) = \tanh(x)$ yields

$$\Lambda(\psi = \varphi + u + iv) - \Lambda(\varphi) = Q_+(u) + Q_-(v) + R(u, v),$$

where $Q_+(u) = (L_+ u, u)_{L^2}$, $Q_-(v) = (L_- v, v)_{L^2}$, and

$$R(u, v) = \int [(2\varphi u + u^2 + v^2)^2 - 4\varphi^2 u^2] dx$$

Main result 2: stability of the black soliton

Expanding the energy functional

$$\Lambda(\psi) := \int [|\psi_x|^2 + (1 - |\psi|^2)^2] dx$$

at the black soliton $\varphi(x) = \tanh(x)$ yields

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Since $\Lambda(\psi)$ conserves in time, the black soliton is energetically stable in a Banach space X if

$$\Lambda(\psi) - \Lambda(\varphi) \geq C(\|u\|_X^2 + \|v\|_X^2) - C(\|u\|_X^3 + \|v\|_X^3).$$

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However, the obstacles arise due to nonzero boundary conditions

- $L_- = -\partial_x^2 - 2\mathrm{sech}^2(x)$ is not coercive in $H^1(\mathbb{R})$
- $R(u, v)$ is not cubic if $(u, v) \notin H^1(\mathbb{R})$.

Main result 2: stability of the black soliton

For the NLS equation, this was corrected in [Gravejat–Smets, 2015]

$$\Lambda(\psi = \varphi + u + iv) - \Lambda(\varphi) = Q_-(u) + Q_-(v) + \|\eta\|_{L^2}^2$$

where $Q_-(v) = (L_- v, v)_{L^2}$ and $\eta := |\psi|^2 - \varphi^2 = 2\varphi u + u^2 + v^2$. The distance for perturbations was chosen to be

$$\mathcal{D}_X(\psi_1, \psi_2) := \sqrt{\|\psi_1' - \psi_2'\|_{L^2}^2 + \||\psi_1|^2 - |\psi_2|^2\|_{L^2}^2 + \|\psi_1 - \psi_2\|_{\mathcal{H}}^2}.$$

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For the NLS with IDD, we have several advantages:

- $\mathcal{H}$ appears naturally in the time evolution
- $Q_-(u)$ and $Q_-(v)$ are coercive in $\mathcal{H}$
- $u \in \mathcal{H}$ satisfies orthogonality $(\varphi', u)_{\mathcal{H}} = (\varphi, u)_{\mathcal{H}} = 0$
- $v \in \mathcal{H}$ satisfies orthogonality $(\varphi', v)_{\mathcal{H}} = (\varphi, v)_{\mathcal{H}} = 0$

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For the four orthogonality conditions, we use the decomposition

$$\psi(t,x) = e^{i\theta(t)} \left[U_{c(t),\omega(t)}(x + \zeta(t)) + u(t, x + \zeta(t)) + iv(t, x + \zeta(t)) \right],$$

where the additional parameter ω is due to the scaling invariance $\psi(t,x) \mapsto \psi(\omega^2 t, \omega x)$ of the NLS equation $i(1 - |\psi|^2)\psi_t + \psi_{xx} = 0$.

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Theorem

Assume that the initial-value problem is well-posed in X with the distance $\mathcal{D}_X$ and the values of $M(\psi)$, $E(\psi)$, and $P(\psi)$ are conserved in the time evolution. Then, the black soliton is orbitally stable in X.

Main result 3: persistence of the black soliton

Adding a small decaying potential yields

$$i(1-|\psi|^2)\psi_t + \psi_{xx} = \varepsilon V(x)\psi$$

The black soliton persists for $\varepsilon \neq 0$ at a non-degenerate extremal point of the effective potential

$$\mathcal{V}(s) := \int_{\mathbb{R}} V(x+s)\mathrm{sech}^2(x)dx.$$

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For the NLS equation, it was found in [P–Kevrekidis, ZAMP (2008)]

- ▷ Pinning at the maximum of $\mathcal{V}(s)$ is unstable with a pair of real eigenvalues
- ▷ Pinning at the minimum of $\mathcal{V}(s)$ is unstable with a quartet of complex eigenvalues.

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For the NLS with IDD, the continuous spectrum is absent, so that

- Pinning at the maximum of $\mathcal{V}(s)$ is unstable with a pair of real eigenvalues
- Pinning at the minimum of $\mathcal{V}(s)$ is stable with a pair of imaginary eigenvalues of negative energy.

Summary on dark solitons

We considered NLS equation with intensity-dependent dispersion

$$i(1 - |\psi|^2)\psi_t + \psi_{xx} = 0.$$

- Linearization at the black soliton consists of isolated eigenvalues
- Perturbations near the black soliton are controlled by the conserved energy, mass, and momentum.
- Stable black solitons are pinned to the minima of small decaying potentials.
- Well-posedness of the model is opened for further studies.

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Happy birthday, Dimitri!