# Solitary waves under intensity-dependent dispersion

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# Classification of solitary waves

Bright soliton $\psi(t,x) = e^{it}\mathrm{sech}(x)$
of the focusing NLS equation

$$i\partial_t\psi + \partial_x^2\psi + 2|\psi|^2\psi = 0$$

with $|\psi(t,x)| \to 0$ as $|x| \to \infty$

Dark soliton $\psi(t,x) = e^{-2it}\tanh(x)$
of the defocusing NLS equation

$$i\partial_t\psi + \partial_x^2\psi - 2|\psi|^2\psi = 0$$

with $|\psi(t,x)| \to 1$ as $|x| \to \infty$

# NLS with intensity-dependent dispersion

The NLS equation realizes a balance between nonlinearity and dispersion for propagation of nonlinear dispersive waves.

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Taking into account higher-order nonlinearity and dispersion gives an extended version of the NLS equation:

$$i\psi_t + \psi_{xx} + |\psi|^2\psi + ic_1\psi_{xxx} + ic_2|\psi|^2\psi_x + ic_3(|\psi|^2\psi)_x + c_4|\psi|^4\psi = 0.$$

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We study different NLS models where the dispersion coefficient depends on the wave intensity:

$$i\psi_t + (1 - |\psi|^2)\psi_{xx} = 0 \quad \text{or} \quad i\psi_t + (1 - |\psi|^2)^{-1}\psi_{xx} = 0.$$

C.Y. Lin, J.H. Chang, G. Kurizki, and R.K. Lee, Optics Letters **45** (2020), 1471–1474

# NLS with intensity-dependent dispersion

For NLS-IDD,

$$i\psi_t + (1 - |\psi|^2)\psi_{xx} = 0, \qquad \text{(NLS-IDD)}$$

two formal conserved quantities exist:

$$Q(\psi) = -\int_{\mathbb{R}} \log|1 - |\psi|^2| dx$$

and

$$E(\psi) = \int_{\mathbb{R}} |\psi_x|^2 dx.$$

Standing waves have the form $\psi(x,t) = e^{i\omega t}u(x)$ with $(\omega, u)$ satisfying

$$\omega u(x) = (1 - u^2)u''(x).$$

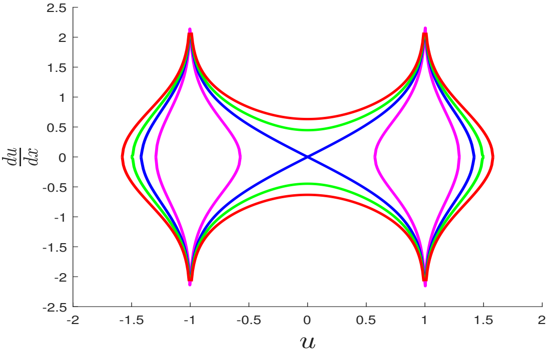
Solitary waves with $u(x) \to 0$ as $|x| \to \infty$ exist only if $\omega > 0$, in which case $\omega$ can be scaled out by $u(x) \mapsto u(\sqrt{\omega}x)$.

# Phase plane portrait

Equation $(1-u^2)u'' = u$ is integrable with the first invariant:

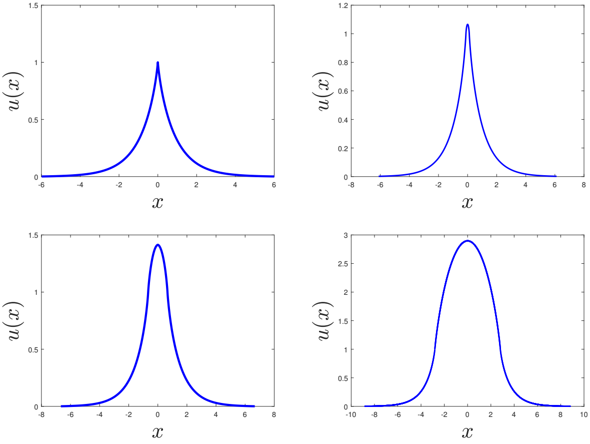
$$\frac{1}{2}\left(\frac{du}{dx}\right)^2 + \frac{1}{2}\log|1-u^2| = C,$$

where $C$ is constant. Bright solitons are singular at $u = \pm 1$.



# Possible solitary waves

Gluing the stable and unstable curves with another integral curves give a one-parameter family of single-humped solitary waves:




Top left: “cusped soliton". Others: “bell-shaped solitons".

# Questions on existence and stability of these solitary waves

- In what space (in what sense) do they exist?
- What is the nature of singularity at $u = \pm 1$?
- Can these solutions be characterized variationally?

# Existence result

## Definition

We say that $u \in H^1(\mathbb{R})$ is a weak solution of the differential equation $u = (1 - u^2)u''$ if it satisfies the following equation

$$\langle u, \varphi \rangle + \langle (1 - u^2)u', \varphi' \rangle - 2\langle u(u')^2, \varphi \rangle = 0, \quad \text{for every } \varphi \in H^1(\mathbb{R}),$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(\mathbb{R})$.

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where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(\mathbb{R})$.

## Theorem (Ross–Kevrekidis–P, Q.Appl.Math. 79 (2021) 641)

*There exists a one-parameter continuous family of weak, positive, and single-humped solutions of $u = (1 - u^2)u''$ parametrized by $C$.*

What is needed for the proof beyond the phase plane analysis:

- ▷ $u \in H^1(\mathbb{R})$;
- ▷ $\lim_{x \to x_0} (1 - u^2(x))u'(x) = 0$ for each $x_0$ where $u(x_0) = 1$.

# Nature of singularity at $u = 1$

It follows from the first invariant

$$\frac{1}{2}\left(\frac{du}{dx}\right)^2 + \frac{1}{2}\log|1-u^2| = C,$$

that the cusped soliton is defined by the implicit function

$$|x| = \int_u^1 \frac{d\xi}{\sqrt{-\log(1-\xi^2)}}, \quad u \in (0,1).$$

Asymptotic analysis gives as $|x| \to 0$:

$$u(x) = 1 - |x|\sqrt{\log(1/|x|)}\left[1 + \mathcal{O}\left(\frac{\log\log(1/|x|)}{\log(1/|x|)}\right)\right].$$

[Alfimov–Korobeinikov–Lustri–P, Nonlinearity 32 (2019) 3445]

Hence, $u'(x) \sim \sqrt{\log(1/|x|)}$ and $(1-u^2)u'(x) \sim |x|\log(1/|x|)$.

# Towards the stability result

Recall the conserved quantities:

$$Q(\psi) = -\int_{\mathbb{R}} \log|1 - |\psi|^2| dx, \quad E(\psi) = \int_{\mathbb{R}} |\psi_x|^2 dx.$$

Solitary wave $\psi(x,t) = u(x)e^{i\omega t}$ is a critical point of the action

$$\Lambda_\omega(u) = E(u) + \omega Q(u),$$

however, the formal expansion yields

$$\begin{aligned} \Lambda_\omega(u+\varphi) - \Lambda_\omega(u) &= 2\langle u', \varphi' \rangle + 2\omega \langle (1-u^2)^{-1} u, \varphi \rangle \\ &\quad + \mathcal{O}(\|\varphi'\|_{L^2}^2 + \|(1-u^2)^{-1}\varphi\|_{L^2 \cap L^\infty}^2), \end{aligned}$$

which is not compatible with the definition of weak solutions:

$$u \in H^1(\mathbb{R}): \quad \omega\langle u, \varphi \rangle + \langle (1-u^2)u', \varphi' \rangle - 2\langle u(u')^2, \varphi \rangle = 0,$$

for every $\varphi \in H^1(\mathbb{R})$.

# New definition of weak solutions

## Definition

Fix $L > 0$ and define

$$X_L := \left\{ u \in H^1(\mathbb{R}) : \; u(x) > 1, \; x \in (-L, L) \text{ and } u(x) \leq 1, \; |x| \geq L \right\}.$$

Pick $u_L \in X_L$ satisfying

$$\lim_{|x| \to L} \frac{u_L(x) - 1}{(L - |x|)\sqrt{|\log|L - |x|||}} = 1.$$

We say that $u \in X_L \subset H^1(\mathbb{R})$ is a weak solution if it satisfies the following equation

$$\langle u', \varphi' \rangle + \omega \langle (1 - u^2)^{-1} u, \varphi \rangle = 0, \quad \text{for every } \varphi \in H^1_L,$$

where $H^1_L := \left\{ \varphi \in H^1(\mathbb{R}) : \;\; (1 - u_L^2)^{-1} \varphi \in L^2(\mathbb{R}) \cap L^\infty(\mathbb{R}) \right\}$.

# Stability result

**Theorem (P–Ross–Kevrekidis, J. Phys. A 54 (2021) 445701)**

*For every $\mu > 0$ and $L > 0$, there exists a unique minimizer of the constrained variational problem*

$$\mathcal{Q}_{\mu,L} := \inf_{u \in X_L} \{Q(u) : \quad E(u) = \mu\}.$$

What is needed for the proof beyond the expansion of $\Lambda_\omega$ in $X_L$:

- ▷ Monotonicity of mappings $C \mapsto E(u_C)$ and $C \mapsto \ell_C$, where $2\ell_C$ is the length of the bell head;
- ▷ Scaling transformation;
- ▷ Convexity of action $\Lambda_{\omega=1}$ at $u_C$.

# Monotonicity of mappings $C \mapsto E(u_C)$ and $C \mapsto \ell_C$

It follows from $(u')^2 + \log|1-u^2| = 2C$ that

$$E(u_C) = E(u_{\text{cusp}}) + 2\int_1^{\sqrt{1+e^{2C}}} \sqrt{2C - \log(u^2-1)}\,du$$

and

$$\ell_C = \int_1^{\sqrt{1+e^{2C}}} \frac{du}{\sqrt{2C - \log(u^2-1)}}$$

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$\frac{dE(u_C)}{dC} > 0$ follows from

$$\frac{dE(u_C)}{dC} = 2\int_1^{\sqrt{1+e^{2C}}} \frac{du}{\sqrt{2C - \log(u^2-1)}} = 2\ell_C.$$

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$\frac{d\ell_C}{dC} > 0$ follows from a longer computation, where we use **the period function** for periodic orbits on the phase plane.

# Monotonicity of mappings $C \mapsto E(u_C)$ and $C \mapsto \ell_C$

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and

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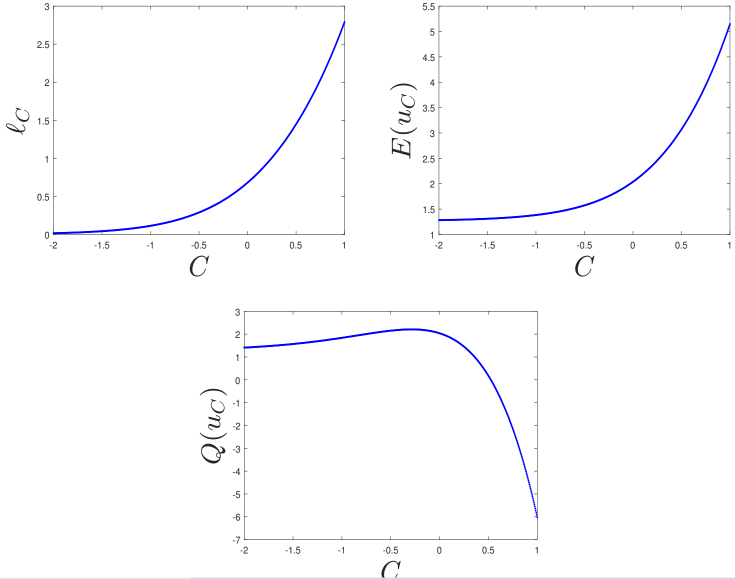
The mapping $C \mapsto Q(u_C)$ is non-monotone.

# Numerical illustrations of mappings $C \mapsto \ell_C, E(u_C), Q(u_C)$



# Scaling transformation

The variational problem for $\mu > 0$ and $L > 0$:

$$\mathcal{Q}_{\mu,L} := \inf_{u \in X_L} \{Q(u) : \quad E(u) = \mu\},$$

is associated with the Euler–Lagrange equation $\omega u = (1 - u^2)u''$.

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Let $u_C$ be a solution of $u = (1 - u^2)u''$. Then, $u_\omega(x) = u_C(\sqrt{\omega}x)$ is a solution of the Euler–Lagrange equation so that

$$Q(u_\omega) = \frac{1}{\sqrt{\omega}} Q(u_C), \quad E(u_\omega) = \sqrt{\omega} E(u_C)$$

and

$$L = \frac{1}{\sqrt{\omega}} \ell_C, \quad \mu = \sqrt{\omega} E(u_C).$$

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Transformation $(\omega, C) \mapsto (\mu, L)$ is invertible because the Jacobian is

$$\begin{vmatrix} \frac{\partial \mu}{\partial \omega} & \frac{\partial \mu}{\partial C} \\ \frac{\partial L}{\partial \omega} & \frac{\partial L}{\partial C} \end{vmatrix} = \frac{1}{2\omega} \left[ E(u_C) \frac{d\ell_C}{dC} + \ell_C \frac{dE(u_C)}{dC} \right] > 0.$$

Hence the mapping $(\omega, C) \mapsto (\mu, L)$ is invertible and there exists a unique $C = C_{\mu,L}$ for every $\mu > 0$ and $L > 0$. In fact, $\ell_C E(u_C) = L\mu$.

# Convexity of action $\Lambda_\omega$

Let $v + iw$ with real $v, w \in H^1_{\ell_C} \subset H^1(\mathbb{R})$ be a perturbation to $u_C$.
Then, the action is expanded as

$$\Lambda_{\omega=1}(u_C + v + iw) = \Lambda_{\omega=1}(u_C) + Q_+(v) + Q_-(w) + R(v,w),$$

where $R(v,w)$ is the remainder term

$$R(v,w) = \int_{\mathbb{R}} \left[ \log\left(1 - \frac{2u_C v + v^2 + w^2}{1 - u_C^2}\right) + \frac{2u_C v}{1-u_C^2} + \frac{(1+u_C^2)v^2}{(1-u_C^2)^2} + \frac{w^2}{1-u_C^2}\right] dx.$$

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$R(v, w)$ is cubic with respect to perturbation:

$$|R(v, w)| \leq C\|(1 - u_C^2)^{-1} v\|^3_{L^2 \cap L^\infty} + C\|(1 - u_C^2)^{-1} w\|^3_{L^2 \cap L^\infty},$$

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$$\Lambda_{\omega=1}(u_C + v + iw) = \Lambda_{\omega=1}(u_C) + Q_+(v) + Q_-(w) + R(v, w),$$

whereas $Q_+$ and $Q_-$ are the quadratic forms:

$$Q_+(v) = \int_{\mathbb{R}} \left[ (v_x)^2 + \frac{(1+u_C^2)v^2}{(1-u_C^2)^2} \right] dx, Q_-(w) = \int_{\mathbb{R}} \left[ (w_x)^2 + \frac{w^2}{1-u_C^2} \right] dx,$$

The quadratic forms are coercive and bounded as

$$Q_\pm(v) \geq \|v\|^2_{H^1}, \quad Q_\pm(v) \leq C_\pm \left( \|v'\|^2_{L^2} + \|(1-u_C^2)^{-1} v\|^2_{L^2} \right)$$

Hence $u_{C_{\mu L}}$ is a minimizer of $Q(u)$ in $X_L$ for fixed $L > 0$ and $\mu > 0$.

# Summary on bright solitons

We considered NLS equation with intensity-dependent dispersion

$$i\psi_t + (1 - |\psi|^2)\psi_{xx} = 0.$$

- Continuum of singular solitary waves exists $\psi(x,t) = u_C(x)e^{it}$.
- Each solitary wave can be characterized as a minimizer of mass for fixed energy and fixed distance between two singularities.
- Well-posedness of the model is opened for further studies.

# Another NLS with intensity-dependent dispersion

For another NLS-IDD,

$$i(1-|\psi|^2)\psi_t + \psi_{xx} = 0, \qquad \text{(NLS-IDD)}$$

transformation $\psi(x,t) = u(x,t)e^{2it}$ recovers the defocusing NLS

$$i(1-|u|^2)u_t + u_{xx} + 2(1-|u|^2)u = 0,$$

which admit the black soliton in the form $u(x) = \tanh(x)$.

Dark solitons $u(t,x) = U_c(x-2ct)$ are found from

$$U_c'' - 2ic(1-|U_c|^2)U_c' + 2(1-|U_c|^2)U_c = 0,$$

for any $c \in \mathbb{R}$.

# Time evolution

Solutions can be considered in the set $\mathcal{F}$,

$$\mathcal{F} := \{ f \in L^\infty(\mathbb{R}) : \quad |f(x)| < 1, \;\; x \in \mathbb{R}, \;\; |f(x)| \to 1 \;\; \text{as} \;\; |x| \to \infty \} .$$

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Conserved quantities of mass and energy

$$M(\psi) = \int (1 - |\psi|^2)^2 dx, \quad E(\psi) = \int |\psi_x|^2 dx$$

and the momentum

$$P(\psi) = \frac{1}{2i} \int \frac{(1 - |\psi|^2)^2}{|\psi|^2} (\bar{\psi}\psi_x - \bar{\psi}_x \psi) dx.$$

Conservation is proven for $\psi(t,x) = e^{i\theta_\pm}(1 + \mathcal{O}(e^{-\alpha_\pm |x|})), x \to \pm\infty$.

# Main result 1: linearization at the black soliton

Using the decomposition $\psi(t,x) = e^{-2it}[\varphi(x) + u(t,x) + iv(t,x)]$, where $\varphi(x) = \tanh(x)$ and $u + iv$ is the perturbation, we obtain the linearized equations of motion

$$(1-\varphi^2)u_t = L_- v, \quad (1-\varphi^2)v_t = -L_+ u,$$

where $L_+ = -\partial_x^2 + 4 - 6\text{sech}^2(x)$ and $L_- = -\partial_x^2 - 2\text{sech}^2(x)$ are the same as in the NLS equation.

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The spectral problem

$$L_- v = \lambda(1-\varphi^2)u, \quad L_+ u = -\lambda(1-\varphi^2)v$$

is defined in the Hilbert space $\mathcal{H}$ with the inner product

$$(f,g)_{\mathcal{H}} := \int (1-\varphi^2)\bar{f} g dx = \int \mathrm{sech}^2(x)\bar{f}(x)g(x)dx.$$

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## Theorem (P–Plum, 2022)

- *The spectrum of $L_+$ in $\mathcal{H}$ consists of simple eigenvalues* $\mu_n = n(n+5)$, $n \geq 0$.
- *The spectrum of $L_-$ in $\mathcal{H}$ consists of simple eigenvalues* $\nu_n = n(n+1) - 2$, $n \geq 0$.
- *The spectrum of the stability problem in $\mathcal{H} \times \mathcal{H}$ consists of pairs of isolated eigenvalues $\{\pm i\omega_1, \pm i\omega_2, \cdots\}$ and zero eigenvalue.*

# Main result 2: energetic stability of the black soliton

Expanding the energy functional

$$\Lambda(\psi) := \int [|\psi_x|^2 + (1 - |\psi|^2)^2] dx$$

at the black soliton $\varphi(x) = \tanh(x)$ yields

$$\Lambda(\psi = \varphi + u + iv) - \Lambda(\varphi) = Q_+(u) + Q_-(v) + R(u,v),$$

where $Q_+(u) = (L_+ u, u)_{L^2}$, $Q_-(v) = (L_- v, v)_{L^2}$, and

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Black soliton is energetically stable in a Banach space $X$ if

$$\Lambda(\psi) - \Lambda(\varphi) \geq C(\|u\|_X^2 + \|v\|_X^2) - C(\|u\|_X^3 + \|v\|_X^3).$$

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However, two obstacles arise due to nonzero boundary conditions

- $L_- = -\partial_x^2 - 2\text{sech}^2(x)$ is not coercive in $H^1(\mathbb{R})$
- $R(u,v)$ is not cubic if $(u,v) \notin H^1(\mathbb{R})$.

# Main result 2: energetic stability of the black soliton

For the cubic NLS, this was corrected in [Gravejat–Smets, 2015]

$$\Lambda(\psi = \varphi + u + iv) - \Lambda(\varphi) = Q_-(u) + Q_-(v) + \|\eta\|_{L^2}^2$$

where $Q_-(v) = (L_- v, v)_{L^2}$ and $\eta := |\psi|^2 - \varphi^2 = 2\varphi u + u^2 + v^2$. The distance for perturbations in Banach space $X$ was chosen to be

$$\mathcal{D}_X(\psi_1, \psi_2) := \sqrt{\|\psi_1' - \psi_2'\|_{L^2}^2 + \||\psi_1|^2 - |\psi_2|^2\|_{L^2}^2 + \|\psi_1 - \psi_2\|_{\mathcal{H}}^2}.$$

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For the NLS–IDD, we have several advantages:

- $\mathcal{H}$ appears naturally in the time evolution
- $Q_-(u)$ and $Q_-(v)$ are coercive in $\mathcal{H}$ if
  - $u \in \mathcal{H}$ satisfies orthogonality $(\varphi', u)_{\mathcal{H}} = (\varphi, u)_{\mathcal{H}} = 0$
  - $v \in \mathcal{H}$ satisfies orthogonality $(\varphi', v)_{\mathcal{H}} = (\varphi, v)_{\mathcal{H}} = 0$

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$$\mathcal{D}_X(\psi_1, \psi_2) := \sqrt{\|\psi_1' - \psi_2'\|_{L^2}^2 + \||\psi_1|^2 - |\psi_2|^2\|_{L^2}^2 + \|\psi_1 - \psi_2\|_{\mathcal{H}}^2}.$$

For the four orthogonality conditions, we use the decomposition

$$\psi(t,x) = e^{i\theta(t)} \left[ U_{c(t),\omega(t)}(x + \zeta(t)) + u(t, x + \zeta(t)) + iv(t, x + \zeta(t)) \right],$$

where the additional parameter $\omega$ is due to the scaling invariance $\psi(t,x) \mapsto \psi(\omega^2 t, \omega x)$ of the NLS equation $i(1 - |\psi|^2)\psi_t + \psi_{xx} = 0$.

# Main result 2: energetic stability of the black soliton

For the cubic NLS, this was corrected in [Gravejat–Smets, 2015]

$$\Lambda(\psi = \varphi + u + iv) - \Lambda(\varphi) = Q_-(u) + Q_-(v) + \|\eta\|^2_{L^2}$$

where $Q_-(v) = (L_- v, v)_{L^2}$ and $\eta := |\psi|^2 - \varphi^2 = 2\varphi u + u^2 + v^2$. The distance for perturbations in Banach space $X$ was chosen to be

$$\mathcal{D}_X(\psi_1, \psi_2) := \sqrt{\|\psi_1' - \psi_2'\|^2_{L^2} + \||\psi_1|^2 - |\psi_2|^2\|^2_{L^2} + \|\psi_1 - \psi_2\|^2_{\mathcal{H}}}.$$

### Theorem (P–Plum, 2022)

*Assume that the initial-value problem is well-posed in $X$ with the distance $\mathcal{D}_X$ and the values of $M(\psi)$, $E(\psi)$, and $P(\psi)$ are conserved in the time evolution. Then, the black soliton is orbitally stable in $X$.*

# Summary on dark solitons

We considered NLS equation with intensity-dependent dispersion

$$i(1 - |\psi|^2)\psi_t + \psi_{xx} = 0.$$

- Linearization at the black soliton consists of isolated eigenvalues
- Perturbations near the black soliton are controlled by the conserved energy, mass, and momentum.
- Well-posedness of the model is opened for further studies.