Asymptotic stability of discrete solitons

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The problem

The discrete nonlinear Schrödinger (DNLS) equation

$$i\dot{u}_n + \Delta_d u_n + \sigma |u_n|^2 u_n = 0, \quad n \in \mathbb{Z}^d.$$

Localized modes (time-periodic space-localized solutions) are of the form $u_n(t) = \phi_n e^{-i\omega t}$, where $\omega \in \mathbb{R}$ and $\{\phi_n\}_{n \in \mathbb{Z}^d}$ satisfies

$$(\omega + \Delta_d)\,\phi_n + \sigma |\phi_n|^2 \phi_n = 0, \quad n \in \mathbb{Z}^d.$$

Main Question: If a localized mode is orbitally stable, is it also asymptotically stable due to dispersive radiation?

- P. Kevrekidis, D. Pelinovsky, and A. Stefanov, arXiv:0810.1778
- S. Cuccagna, M. Tarulli, arXiv:0808.2024

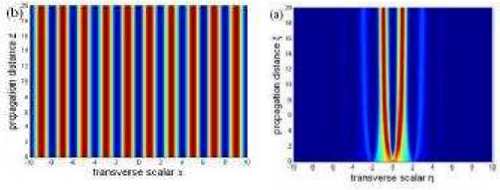
Physical contexts

The DNLS equation arises in the modeling of density waves in Bose–Einstein condensates as the Gross–Pitaevskii equation

$$iu_t = -\nabla^2 u + V(x)u + \sigma|u|^2 u$$

with a bounded periodic potential $V(x) = V(x + 2\pi)$ reduces asymptotically to the DNLS equation in a tight-binding approximation.

Another context of the DNLS equation is the coupled waveguide arrays in nonlinear optics and photorefractive crystals.

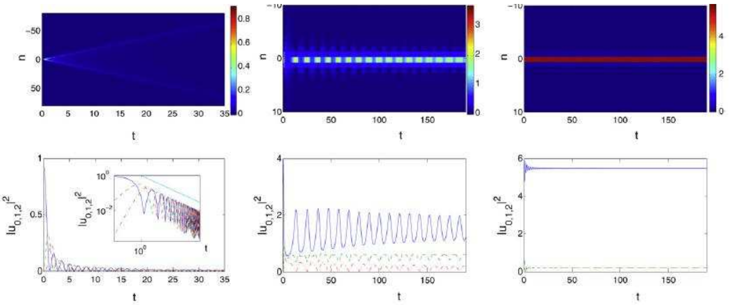


Numerical simulations

P. Kevrekidis et al., Physics Letters A **372**, 2237 (2008)

$$\text{(1D)} \quad i\dot{u}_n + u_{n+1} - 2u_n + u_{n-1} + |u_n|^2 u_n = 0$$

$u_n(0) = A\delta_{n,0}$ with $A = 1$ (left), $A = 2$ (middle), and $A = 2.5$ (right).

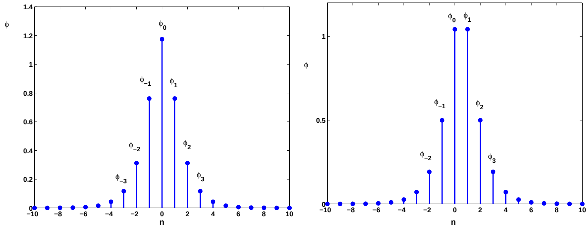


Asymptotic stability of localized modes

Given a time-periodic space-localized solution $\phi_n e^{-i\omega t}$ of the DNLS equation, the stability can be considered in the following three senses:

- Linearized (or spectral) stability
- Nonlinear orbital stability
- Asymptotic stability

Stability depends on ϕ_n. In what follows, we consider single-humped on-site discrete solitons, which are known to be spectrally stable.



Left: on-site soliton. Right: inter-site soliton.

More general formulation

Let us consider the 1D DNLS equation in the form

$$i\dot{u}_n = (-\Delta + V_n)u_n + \sigma |u_n|^{2p} u_n, \quad n \in \mathbb{Z},$$

where $\sigma = \pm 1$, $p \geq 1$ (an integer), and $V \in l^\infty$.

Assumption: V supports exactly one negative eigenvalue $\omega_0 < 0$ of $H = -\Delta + V$ with an eigenvector $\psi_0 \in l^2$ (normalized by $\|\psi_0\|_{l^2} = 1$).

For instance, if $V_n = -\delta_{n,0}$, the assumption is satisfied with

$$(\psi_0)_n = e^{-\kappa |n|}, \quad n \in \mathbb{Z},$$

where $\kappa = \mathrm{arcsinh}(2^{-1})$ and $\omega_0 = 2 - \sqrt{5} < 0$.

Global well-posedness

Global well-posedness: For any $\mathbf{u}_0 \in l^2_\sigma$, $\sigma \geq 0$, there exists a unique solution $\mathbf{u}(t) \in C^1(\mathbb{R}_+, l^2_\sigma)$ s.t. $\mathbf{u}(0) = \mathbf{u}_0$ and $\mathbf{u}(t)$ depends continuously on $\mathbf{u}_0$.

Local existence follows from the Picard iterations applied to

$$u_n(t) = u_n(0) - i \int_0^t \left[(-\Delta + V_n) u_n(t') + \sigma |u_n(t')|^{2p} u_n(t') \right] dt'$$

in space $C([0,T], l^2_\sigma)$. To show that $T = \infty$, we can use the balance equation

$$i \frac{d}{dt} |u_n|^2 = u_n(\bar{u}_{n+1} + \bar{u}_{n-1}) - \bar{u}_n(u_{n+1} + u_{n-1}),$$

so that

$$\|\mathbf{u}(t)\|^2_{l^2_\sigma} \leq \|\mathbf{u}(0)\|^2_{l^2_\sigma} + C \int_0^t \|\mathbf{u}(t')\|^2_{l^2_\sigma} dt'.$$

By Gronwall's inequality, $\|\mathbf{u}(t)\|^2_{l^2_\sigma}$ is bounded and continuous for any $t \in \mathbb{R}_+$.

Local bifurcation of localized modes

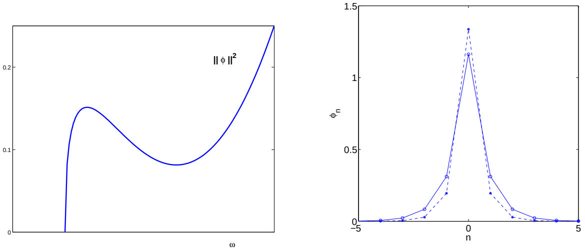
Local bifurcation: Let $\epsilon := \omega - \omega_0$ and $\sigma = +1$. For any $\epsilon \in (0, \epsilon_0)$, where $\epsilon_0 > 0$ is small, there exists a solution $\phi(\omega) \in C([\omega_0, \omega_0 + \epsilon_0), l^2)$ of

$$(-\Delta + V_n)\phi_n(\omega) + \sigma \phi_n^{2p+1}(\omega) = \omega \phi_n(\omega), \quad n \in \mathbb{Z},$$

satisfying

$$\left\| \phi(\omega) - \frac{\epsilon^{\frac{1}{2p}} \psi_0}{\|\psi_0\|_{l^{2p+2}}^{1+\frac{1}{p}}} \right\|_{l^2} \leq C \epsilon^{1+\frac{1}{2p}}.$$

Moreover, $\phi(\omega)$ decays exponentially to zero as $|n| \to \infty$.



Orbital stability

Theorem [Weinstein, 1999]: Let $\sigma = -1$ and $V \equiv 0$. There exists a global minimizer of energy

$$E = \sum_{n \in \mathbb{Z}} |u_{n+1} - u_n|^2 - \frac{1}{p+1} |u_n|^{2p+2}$$

under a fixed $N = \|u\|_{l^2}^2$ for any $p \geq 1$. If $p < 2$, it exists for any $N > 0$, whereas for $p \geq 2$, there is a threshold $N_0 > 0$ so that it exists for $N \geq N_0$.

If $\mathbf{u}(0) \approx \phi(\omega(0))$, then $\mathbf{u}(t)$ remains near $\phi(\omega(t))$ for all $t \in \mathbb{R}_+$. However, the question is if there exists ω_∞ so that $\mathbf{u}(t) \to \phi(\omega_\infty)$ as $t \to \infty$.

Main result

Theorem [Kevrekidis, P., Stefanov, 2008]: Let $\sigma = +1$ and $p \geq 3$. Fix $\epsilon > 0$ and $\delta > 0$ be small and assume that $\omega(0) = \omega_0 + \epsilon$ and

$$\|\mathbf{u}(0) - \phi(\omega_0 + \epsilon)\|_{l^2} \leq \delta \epsilon^{\frac{1}{2p}}.$$

Under some assumptions on V, there exist $\omega_\infty \in (\omega_0, \omega_0 + \epsilon_0)$, $(\omega, \theta) \in C^1(\mathbb{R}_+)$, and $\mathbf{y}(t) = \mathbf{u}(t) - e^{-i\theta(t)}\phi(\omega(t)) \in C^1(\mathbb{R}_+, l^2) \cap L^6(\mathbb{R}_+, l^\infty)$ such that $\mathbf{u}(t)$ solves the DNLS equation and

$$\lim_{t\to\infty} \omega(t) = \omega_\infty, \quad \lim_{t\to\infty} \|\mathbf{u}(t) - e^{-i\theta(t)}\phi(\omega(t))\|_{l^\infty} = 0.$$

Remark: A similar result applies in the focusing case $\sigma = -1$ with the local bifurcation to $\omega < \omega_0$.

Pioneer works on continuous NLS equations are by Soffer,Weinstein (1990,1992,2004), Pillet,Wayne (1997), Yao,Tsai (2002), Mizumachi (2008).

Decomposition of the solution

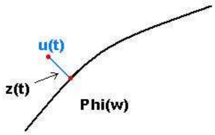
Let

$$\mathbf{u}(t) = e^{-i\theta(t)} \left(\phi(\omega(t)) + \mathbf{z}(t)\right),$$

for some $(\omega, \theta) \in \mathbb{R}^2$. Then, $\mathbf{z}(t) \in C^1(\mathbb{R}_+, l^2_\sigma)$ solves

$$i\dot{\mathbf{z}} = (H - \omega)\mathbf{z} - (\dot{\theta} - \omega)(\phi(\omega) + \mathbf{z}) - i\dot{\omega}\partial_\omega \phi(\omega) + \mathbf{N}(\phi(\omega) + \mathbf{z}) - \mathbf{N}(\phi(\omega)),$$

where $H = -\Delta + V$ and $[\mathbf{N}(\psi)]_n = \sigma |\psi_n|^{2p} \psi_n$.



Question: How to ensure that the decomposition is unique?

Double null space

Linearized time evolution for $\mathbf{z}(t) = \mathbf{v}(t) + i\mathbf{w}(t)$ is defined by the non-self-adjoint eigenvalue problem

$$L_+ \mathbf{v} = -\lambda \mathbf{w}, \quad L_- \mathbf{w} = \lambda \mathbf{v},$$

where

$$L_- = H - \omega + W, \quad L_+ = H - \omega + (2p+1)W,$$

where $W_n = \sigma \phi_n^{2p}(\omega)$.

There exists a double zero eigenvalue with a one-dimensional kernel, isolated from the rest of the spectrum. The generalized kernel is spanned by vectors $(\mathbf{0}, \phi(\omega)), (-\partial_\omega \phi(\omega), \mathbf{0}) \in l^2$ satisfying

$$L_- \phi(\omega) = \mathbf{0}, \qquad L_+ \partial_\omega \phi(\omega) = \phi(\omega).$$

$(\mathbf{v}, \mathbf{w}) \in l^2$ is symplectically orthogonal to the double subspace of the generalized kernel under the conditions

$$\langle \mathbf{v}, \phi(\omega) \rangle = 0, \quad \langle \mathbf{w}, \partial_\omega \phi(\omega) \rangle = 0,$$

where $\langle \mathbf{u}, \mathbf{v} \rangle := \sum_{n \in \mathbb{Z}} u_n \bar{w}_n$.

Symplectic orthogonality

Let us require that

$$\langle \mathrm{Re}\mathbf{z}(t), \psi_1 \rangle = \langle \mathrm{Im}\mathbf{z}(t), \psi_2 \rangle = 0,$$

where

$$\psi_1 = \frac{\phi(\omega)}{\|\phi(\omega)\|_{l^2}}, \quad \psi_2 = \frac{\partial_\omega \phi(\omega)}{\|\partial_\omega \phi(\omega)\|_{l^2}}.$$

Unique decomposition: Fix $\epsilon \in (0, \epsilon_0)$. There exists $\delta > 0$ and $T > 0$ such that any $\mathbf{u}(t) \in C^1([0,T], l^2)$ satisfying

$$\|\mathbf{u}(t) - \phi(\omega_0 + \epsilon)\|_{l^2} \leq \delta \epsilon^{\frac{1}{2p}}, \quad t \in [0,T],$$

can be uniquely decomposed by

$$\mathbf{u}(t) = e^{-i\theta(t)} \left(\phi(\omega(t)) + \mathbf{z}(t)\right),$$

where $(\omega, \theta) \in C^1([0,T], \mathbb{R}^2)$ and $\mathbf{z}(t) \in C^1([0,T], l^2)$ satisfies the symplectic orthogonality conditions. Moreover, there exists $C > 0$ such that

$$|\omega(t) - \omega_0 - \epsilon| \leq C\delta\epsilon, \quad |\theta(t)| \leq C\delta, \quad \|\mathbf{z}(t)\|_{l^2} \leq C\delta\epsilon^{\frac{1}{2p}}, \quad t \in [0,T].$$

Projections

The time-evolution of (ω, θ) satisfies the system

$$\mathbf{A}(\omega, \mathbf{z}) \left[\begin{array}{c} \dot{\omega} \\ \dot{\theta} - \omega \end{array} \right] = \mathbf{f}(\omega, \mathbf{z}),$$

where

$$\mathbf{A}(\omega, \mathbf{z}) = \left[\begin{array}{cc} \langle \partial_\omega \phi(\omega), \psi_1 \rangle - \langle \mathrm{Re} \mathbf{z}, \partial_\omega \psi_1 \rangle & \langle \mathrm{Im} \mathbf{z}, \psi_1 \rangle \\ \langle \mathrm{Im} \mathbf{z}, \partial_\omega \psi_2 \rangle & \langle \phi(\omega) + \mathrm{Re} \mathbf{z}, \psi_2 \rangle \end{array} \right]$$

and

$$\mathbf{f}(\omega, \mathbf{z}) = \left[\begin{array}{l} \langle \mathrm{Im} \mathbf{N}(\phi + \mathbf{z}) - \mathbf{W} \mathbf{z}, \psi_1 \rangle \\ \langle \mathrm{Re} \mathbf{N}(\phi + \mathbf{z}) - \mathbf{N}(\phi) - (2p+1) \mathbf{W} \mathbf{z}, \psi_2 \rangle \end{array} \right].$$

In addition, we recall the time evolution of $\mathbf{z}(t)$ from

$$i \dot{\mathbf{z}} = (H - \omega) \mathbf{z} - (\dot{\theta} - \omega)(\phi(\omega) + \mathbf{z}) - i \dot{\omega} \partial_\omega \phi(\omega) + \mathbf{N}(\phi(\omega) + \mathbf{z}) - \mathbf{N}(\phi(\omega)).$$

Dispersive decay estimates

Assumption: Let $V \in l^1_{2\sigma}$ for a fixed $\sigma > \frac{5}{2}$ and let V is generic in the sense that no solution ψ_0 of equation $H\psi_0 = 0$ exists in $l^2_{-\sigma}$ for $\frac{1}{2} < \sigma \leq \frac{3}{2}$.

Pointwise dispersive decay estimates: There exists a constant $C > 0$ depending on V such that

$$\begin{aligned} \left\| \langle n \rangle^{-\sigma} e^{-itH} P_{a.c.}(H) \mathbf{f} \right\|_{l^2_n} &\leq C(1+t)^{-3/2} \|\langle n \rangle^{\sigma} \mathbf{f}\|_{l^2_n}, \\ \left\| e^{-itH} P_{a.c.}(H) \mathbf{f} \right\|_{l^\infty_n} &\leq C(1+t)^{-1/3} \|\mathbf{f}\|_{l^1_n}, \end{aligned}$$

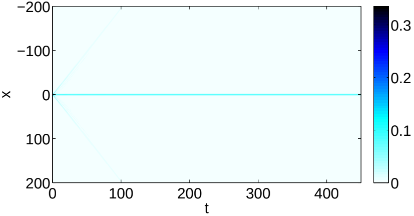
for all $t \in \mathbb{R}_+$.

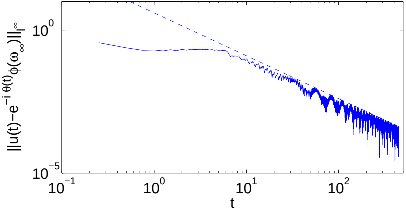
Discrete Strichartz estimates: There exists a constant $C > 0$ such that

$$\begin{aligned} \left\| e^{-itH} P_{a.c.}(H) \mathbf{f} \right\|_{L^6_t l^\infty_n \cap L^\infty_t l^2_n} &\leq C \|\mathbf{f}\|_{l^2_n}, \\ \left\| \int_0^t e^{-i(t-s)H} P_{a.c.}(H) \mathbf{g}(s) ds \right\|_{L^6_t l^\infty_n \cap L^\infty_t l^2_n} &\leq C \|\mathbf{g}\|_{L^1_t l^2_n}. \end{aligned}$$

Numerical results

For any $p = 1, 2, 3$, it was found that $\|\mathbf{y}(t)\|_{l^\infty} = O(t^{-3/2})$ as $t \to \infty$.





Further results:

- Cuccagna' 09: periodic oscillations of discrete solitons with V supporting two eigenvalues
- Stefanov' 09: pushing analysis to $p = 2$