Broad Band Solitons in a Periodic and Nonlinear Maxwell System

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Shocks and Spatial Periodicity

Spatially Homogeneous Quasilinear Hyperbolic System

$$\partial_t \mathbf{v} + \partial_x \mathbf{f}(\mathbf{v}) = 0$$

Smooth data generates typically a shock wave in finite time (Lax 64)

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Spatially Periodic Quasilinear Hyperbolic System

$$\partial_t \mathbf{v} + \partial_x \mathbf{f}(x, \mathbf{v}) = 0, \quad \mathbf{f}(x + 2\pi, \mathbf{v}) = \mathbf{f}(x, \mathbf{v})$$

Can spatial periodicity stabilize shock formation?

Regularizing Shocks

- Diffusive regularization:

$$v_t + v v_x = \mu v_{xx}$$

- Dispersive regularization:

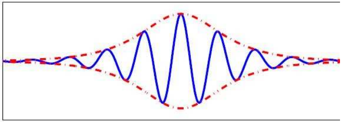
$$v_t + v v_x = \alpha v_{xxx}.$$

- Dispersion from spatial periodicity (Maxwell Model):

$$\partial_t^2 \left(n^2(z) E + \chi E^3 \right) = \partial_z^2 E,$$

where $n(z + 2\pi) = n(z)$ is the refractive index of the periodic media.
  - Does this model display wave breaking (shocks)?
  - Does this model admit stable localized states (solitons)?

Maxwell & Coupled Mode Equations



Periodic Nonlinear Maxwell Equation

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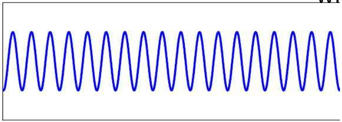
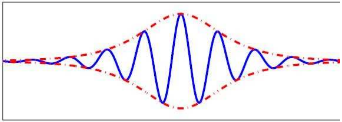


$$n^2(z) = 1 + \epsilon \sum_{p \in \mathbb{Z}\backslash\{0\}} N_p e^{ipz}, \quad \epsilon \ll 1.$$

Maxwell & Coupled Mode Equations



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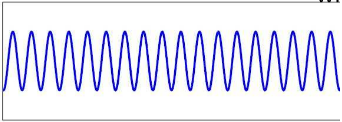

$$n^2(z) = 1 + \epsilon \sum_{p \in \mathbb{Z} \backslash \{0\}} N_p e^{ipz}, \quad \epsilon \ll 1.$$

Two-wave approximation of small-amplitude resonant waves

$$E \approx \epsilon^{1/2} \left(\mathcal{E}^+(\epsilon z, \epsilon t) e^{i(z-t)} + \mathcal{E}^-(\epsilon z, \epsilon t) e^{-i(z+t)} \right)$$

yields the Nonlinear Coupled Mode Equations (NLCME) for $\mathcal{E}^{\pm}(Z,T)$ in slow variables $Z = \epsilon z$ and $T = \epsilon t$.

Properties of the NLCME

The Nonlinear Coupled Mode Equations (NLCME)

$$\partial_T \mathcal{E}^+ + \partial_Z \mathcal{E}^+ = iN_2 \mathcal{E}^- + i\Gamma \left(|\mathcal{E}^+|^2 + 2|\mathcal{E}^-|^2 \right) \mathcal{E}^+,$$
$$\partial_T \mathcal{E}^- - \partial_Z \mathcal{E}^- = i\bar{N}_2 \mathcal{E}^+ + i\Gamma \left(|\mathcal{E}^-|^2 + 2|\mathcal{E}^+|^2 \right) \mathcal{E}^-$$

where $\Gamma = 3\chi/2$.

- Dispersive: $\mathcal{E}^\pm \; e^{i(KZ - \Omega T)}$ with $\Omega^2 = K^2 + |N_2|^2$,
- Possess explicit solitary wave solutions (Aceves–Wabnitz 89),
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- Mathematically inconsistent, because the correction term $\tilde{\mathcal{E}}$,

$$\left(\partial_t^2 - \partial_z^2\right) \tilde{\mathcal{E}} = \left(\mathcal{E}^+\right)^3 e^{3i(z-t)} + \left(\mathcal{E}^-\right)^3 e^{-3i(z+t)} + \dots,$$

  grow linearly in t.

Numerics with Soliton Data

Seed NLCME Soliton $(\mathcal{E}^+, \mathcal{E}^-)$ into Maxwell equations,

$$E(z,t) = \epsilon^{1/2} \left(\mathcal{E}^+(\epsilon z, \epsilon t) e^{i(z-t)} + \mathcal{E}^-(\epsilon z, \epsilon t) e^{-i(z+t)} \right).$$

- No periodic potential:

$$\partial_t^2 \left(E + \chi E^3 \right) = \partial_z^2 E$$

- Small cos-periodic potential:

$$\partial_t^2 \left(E + \epsilon \cos(z) E + \chi E^3 \right) = \partial_z^2 E$$

Side pulses are absent in the NLCME.

Local and global existence

The normalized NLCME:

$$\partial_T \mathcal{E}^+ + \partial_Z \mathcal{E}^+ = i\mathcal{E}^- + i\left(|\mathcal{E}^+|^2 + 2|\mathcal{E}^-|^2\right)\mathcal{E}^+,$$
$$\partial_T \mathcal{E}^- - \partial_Z \mathcal{E}^- = i\mathcal{E}^+ + i\left(|\mathcal{E}^-|^2 + 2|\mathcal{E}^+|^2\right)\mathcal{E}^-.$$

Theorem

Assume the initial data in $H^1(\mathbb{R})$. There exists a unique global solution in $C(\mathbb{R}, H^1(\mathbb{R}))$, which depends continuously on the initial data.

References: Delgado (1978); Goodman-Weinstein-Holmes (2001); Selberg-Tesfahun (2010); Huh (2011); Zhang (2013).

Quick proof of global well-posedness in $H^1(\mathbb{R})$

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- By Gronwall's inequality, we have

$$\|\mathcal{E}_\pm(T)\|_{L^{2p+2}} \leq e^{2|T|}\|\mathcal{E}_\pm(0)\|_{L^{2p+2}}, \quad T \in \mathbb{R},$$

which holds for any $p \geq 0$ including $p \to \infty$.

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  which holds for any $p \geq 0$ including $p \to \infty$.
- This allows to control

$$\frac{d}{dT}\|\partial_Z\mathcal{E}_\pm(T)\|_{L^2}^2 \leq Ce^{8|t|}\|\partial_Z\mathcal{E}_\pm(T)\|_{L^2}^2.$$

Existence of solitary waves

Time-periodic space-localized solutions

$$\mathcal{E}_+(Z,T) = U_\omega(Z) e^{-i\omega T}, \quad \mathcal{E}_-(Z,T) = V_\omega(Z) e^{-i\omega T}$$

are known in the closed analytic form:

$$U_\omega(Z) = i \sin(\gamma) \operatorname{sech}\left[Z \sin\gamma - i\frac{\gamma}{2} \right] = \bar{V}_\omega(Z),$$

where $\omega = \cos(\gamma) \in (-1,1)$.

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- Translations in Z and T can be added as free parameters.
- Constraint $\omega = \cos\gamma \in (-1,1)$ exists because spectrum of linear waves is located for $(-\infty,-1] \cup [1,\infty)$.
- Moving solitons can be obtained from the stationary solitons with a generalized Lorentz transformation.

Stability of solitary waves

- Spectral stability of solitary waves was mainly studied numerically, e.g., by I. Barashenkov (1998), G. Gottwald (2005), M. Chugunova (2006), A. Comech (2012), A. Saxena (2014), P. Kevrekidis (2014).
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For the NLCME, the solitary waves

$$\mathcal{E}_+(Z,T) = U_\omega(Z)e^{-i\omega T}, \quad \mathcal{E}_-(Z,T) = V_\omega(Z)e^{-i\omega T}$$

are **spectrally stable** for $\omega \in (0,1)$ and **unstable** for $\omega \in (-1,0)$.

Back to properties of the NLCME

The Nonlinear Coupled Mode Equations (NLCME)

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$$\partial_T \mathcal{E}^- - \partial_Z \mathcal{E}^- = i\bar{N}_2 \mathcal{E}^+ + i\Gamma \left(\left|\mathcal{E}^-\right|^2 + 2\left|\mathcal{E}^+\right|^2 \right) \mathcal{E}^-$$

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grow linearly in t.

Revised Asymptotic Expansion

Simpson–Weinstein' 2011

Nonlinear and Periodic Maxwell Equation

$$\partial_t^2 \left(E + \epsilon N(z) E + \chi |E|^2 E \right) = \partial_z^2 E.$$

Generalized Ansatz

$$E = \epsilon^{1/2} \left(E^+(z-t, Z, T) + E^-(z+t, Z, T) + \epsilon E^{(1)}(z,t) + \dots \right).$$

Constraint on the Growth of the Correction Term

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \left\| E^{(1)} \right\| dt = 0.$$

Integro-Differential equations for $E^{\pm}(\phi, Z, T)$

Let $N(z) = N(z + 2\pi)$. The correction term is bounded if

$$
\begin{aligned}
(\partial_T + \partial_Z)E^+ = \partial_\phi \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} N(\phi + \theta) E^-(Z, T, \phi + 2\theta) d\theta \right] \\
+ \frac{\Gamma}{3} \partial_\phi \left[(E^+)^3 + 3 \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |E^-(Z, T, \theta)|^2 d\theta \right) E^+ \right], \\
(\partial_T - \partial_Z)E^- = -\partial_\phi \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} N(\phi - \theta) E^+(Z, T, \phi - 2\theta) d\theta \right] \\
- \frac{\Gamma}{3} \partial_\phi \left[(E^-)^3 + 3 \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |E^+(Z, T, \theta)|^2 d\theta \right) E^- \right],
\end{aligned}
$$

where $\Gamma \equiv \frac{3}{2}\chi$.

Extended Nonlinear Coupled Mode Equations (xNLCMEs)

Periodically Varying Index of Refraction

$$N(z) = N(z + 2\pi) \quad \Rightarrow \quad N(z) = \sum N_p e^{ipz}, \quad N_0 = 0$$

Fourier Decomposition

$$E^{\pm}(\phi, Z, T) = \sum E_p^{\pm}(Z, T) e^{ip\phi}.$$

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Fourier Decomposition

$$E^{\pm}(\phi, Z, T) = \sum E_p^{\pm}(Z, T) e^{ip\phi}.$$

Fourier amplitudes satisfy the infinite-dimensional NLCMEs:

$$\partial_T E_p^+ + \partial_Z E_p^+ = \mathrm{i}pN_{2p}E_p^- + \frac{\mathrm{i}p}{3}\left[\sum E_q^+ E_r^+ E_{p-q-r}^+ + 3\left(\sum |E_q^-|^2\right) E_p^+\right]$$

$$\partial_T E_p^- - \partial_Z E_p^- = \mathrm{i}p\bar{N}_{2p}E_p^+ + \frac{\mathrm{i}p}{3}\left[\sum E_q^- E_r^- E_{p-q-r}^- + 3\left(\sum |E_q^+|^2\right) E_p^-\right]$$

Numerics with Soliton Data

Inclusion of third harmonic ($E^{\pm}_{\pm 3}$), resolves side pulses

Questions:

- Do the xNLCMEs admit localized stationary states (solitons)?
- Are localized states robust in the dynamics of the xNLCMEs?

Simplifications:

1. We reduce the system of xNLCMEs near band edges to a system of coupled nonlinear Schrödinger equations.
2. We use the Gaussian trial functions and variational approximations.
3. We truncate the system of equations and perform numerical continuations.

Localized stationary solutions

Stationary decomposition

$$E_p^{\pm}(Z,T) = A_p^{\pm}(Z) e^{-ip\Omega T},$$

Amplitude equations (xNLCMEs)

$$iA_p'(Z) + p\Omega A_p + pN_{2p} B_p + p\frac{\Gamma}{3}\left(3A_p \sum_{q\in\mathbb{Z}} |B_q|^2 + \sum_{q,r\in\mathbb{Z}} A_q A_r A_{p-q-r}\right) = 0,$$

$$-iB_p'(Z) + p\Omega B_p + p\bar{N}_{2p} A_p + p\frac{\Gamma}{3}\left(3B_p \sum_{q\in\mathbb{Z}} |A_q|^2 + \sum_{q,r\in\mathbb{Z}} B_q B_r B_{p-q-r}\right) = 0,$$

Band Edge Approximation

Linear approximation

$$A_p^{\pm}(Z) \sim e^{-|p||Z|\sqrt{|N_{2p}|^2-\Omega^2}}.$$

Exponential decay is provided by the assumption

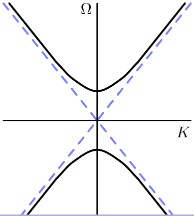
$$N_{2p} = 1 \quad \text{for all} \;\; p \;\; \text{and} \;\; \Omega \in (-1, 1).$$

(Therefore, $N(z)$ is a periodic sequence of Dirac delta-distributions.)

Localized states near a band edge

$$A_p^{\pm}(Z) = \pm \mu U_p(\mu Z) + \mathrm{O}(\mu^2), \;\; \Omega = \sqrt{1-\mu^2}.$$

This expansion allows us to derive coupled nonlinear Schrödinger equations.



The coupled NLS equations

Coupled Stationary Nonlinear Schrödinger Equation

$$U_p''(\zeta) - p^2 U_p + \tfrac{2}{3} p^2 \left(3 U_p \sum |U_q|^2 + \sum U_q U_r U_{p-q-r} \right) = 0,$$

where $\zeta = \mu Z$.

With the Fourier series,

$$U(\theta, \zeta) = \sum_{p \in \mathbb{Z}_{\rm odd}} U_p(\zeta) e^{ip\theta}$$

the system is converted to a scalar equation

$$(\partial_\zeta^2 + \partial_\phi^2) U = \frac{2}{3} \partial_\phi^2 \left[U^3 + 3 \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |U(\zeta, \theta)|^2 d\theta \right) U \right].$$

Justification theorem

Theorem

Assume the existence of a localized state $U \in X^s$ of the NLS equations,

$$X^s \equiv \left\{ U(\zeta,\phi) \in H^s(\mathbb{R} \times \mathbb{T}) : \quad \bar{U}(\zeta,\phi) = U(\zeta,\phi), \right\}, \quad s > 1,$$

satisfying the symmetry $U_p(\zeta) = U_{-p}(-\zeta)$. There exists $\mu_0 > 0$ such that for any $|\mu| < \mu_0$, the xNLCMEs with $\Omega = \sqrt{1-\mu^2}$ admit a unique localized state $A^{\pm} \in X^s$ satisfying the bound

$$\exists C > 0 : \quad \|A^{\pm} \mp \mu U(\mu \cdot, \cdot)\|_{X^s} \leq C\mu^2.$$

- $H^s(\mathbb{R} \times \mathbb{T})$ is a Banach algebra with respect to the pointwise multiplication for any $s > 1$.
- If $U \in X^s$ for $s > 1$, then $U \in L^{\infty}(\mathbb{R} \times \mathbb{T})$ and

$$\lim_{|\zeta| \to \infty} U(\zeta,\phi) = 0, \quad \forall \phi \in \mathbb{T}.$$

Existence of localized stationary states

Coupled NLS equations

$$U_p''(\zeta) - p^2 U_p + \tfrac{2}{3} p^2 \left(3 U_p \sum |U_q|^2 + \sum U_q U_r U_{p-q-r} \right) = 0$$

The main question is to establish the existence of a localized state $U \in X^s$ for $s > 1$ satisfying the symmetry $U_p(\zeta) = U_{-p}(-\zeta)$.

For a scalar NLS equation at $p = \pm 1$, we have the NLS soliton

$$U_{\pm 1} = \frac{1}{\sqrt{3}} \mathrm{sech}(\zeta).$$

Will this solution persist in the system of coupled NLS equations?

Energy arguments

Energy functional is well defined in X^s for any $s \geq 1$.

$$H = \int_{\mathbb{R}} \left[\sum_{p \in \mathbb{Z}} \frac{1}{p^2} |U_p'|^2 - \left(\sum_{p \in \mathbb{Z}} |U_p|^2 \right)^2 - \frac{1}{3} \sum_{p,q,r \in \mathbb{Z}} \bar{U}_p U_q U_r \bar{U}_{q+r-p} \right] d\zeta$$

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Constrained variational problem

$$\text{minimize } H \text{ subject to fixed } N = \int_{\mathbb{R}} \sum |U_p|^2 d\zeta.$$

However, H is unbounded from below, even under the constraint. Let

$$U_p(\zeta) = \lambda_n^{1/2} W(\lambda_n \zeta) \left(\delta_{p,n} + \delta_{p,-n} \right), \quad p \in \mathbb{Z},$$

where $W \in H^1(\mathbb{R})$ is fixed and $\lambda_n = n$, $n \in \mathbb{N}$. Then,
$H = \|W'\|_{L^2}^2 - 6n\|B\|_{L^4}^4 \to -\infty$ as $n \to \infty$.

Rayleigh–Ritz Approximation

Gaussian Ansatz

$$U_p(\zeta) = a_p e^{-b_p \zeta^2}, \quad p \in \mathbb{Z}_{\mathrm{odd}},$$

Reduced Energy

$$H_{\mathrm{G}} = \sum \frac{\sqrt{b_p} a_p^2}{p^2} + \frac{a_p^2}{\sqrt{b_p}} - \frac{a_p^2 a_q^2}{\sqrt{b_p + b_q}} - \frac{\sqrt{2} a_p a_q a_r a_{p-q-r}}{3\sqrt{b_p + b_q + b_r + b_{p-q-r}}}.$$

Euler–Lagrange Equations

$$\nabla_{\mathbf{a}} H_{\mathrm{G}}(\mathbf{a}, \mathbf{b}) = 0, \quad \nabla_{\mathbf{b}} H_{\mathrm{G}}(\mathbf{a}, \mathbf{b}) = 0.$$

Rayleigh–Ritz Approximation, Results

Truncated Solutions of Euler–Lagrange Equations:

No. of Modes	a_1	b_1	a_3	b_3	a_5	b_5
1	0.56060	0.33333	-	-	-	-
2	0.56321	0.33148	-0.13918	3.9413	-	-
3	0.56329	0.33189	-0.14585	3.6287	0.062822	8.5577

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Questions:

- Does the solution converge to a localized state with finite energy H?
- Is the alternating sign between the modes important?
- Does the alternating sign persist with the number of modes?

Reduced Rayleigh–Ritz Approximation

Simplified Gaussian Ansatz

$$U_p(\zeta) = a_p e^{-b_p\zeta^2}, \quad p \in \mathbb{Z}_{\rm odd},$$

with

$$a_p = A(-1)^{(|p|-1)/2}\,|p|^{-\gamma}, \quad b_p = \tfrac{p^2}{3}$$

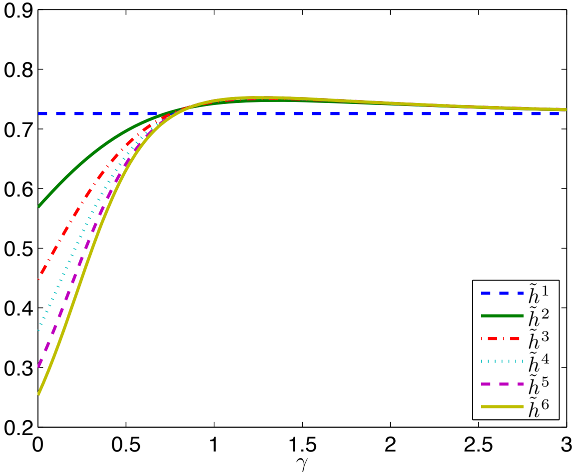
Two Parameter Energy

$$H_{\rm G} \equiv h_G(\gamma, A) = A^2 f(\gamma) - A^4 g(\gamma)$$

At a critical point, this expression simplifies to

$$h_G(\gamma, A(\gamma)) = \frac{f^2(\gamma)}{4g(\gamma)}$$

Reduced Rayleigh–Ritz Approximation, Results



The Gaussian ansatz

$$|U_p(\zeta)| \sim |p|^{-\gamma} e^{-\frac{p^2}{3}\zeta^2},$$

with

$$\gamma_\star \sim 1.26$$

produces

$$U \in X^s, \quad s < \gamma_\star$$

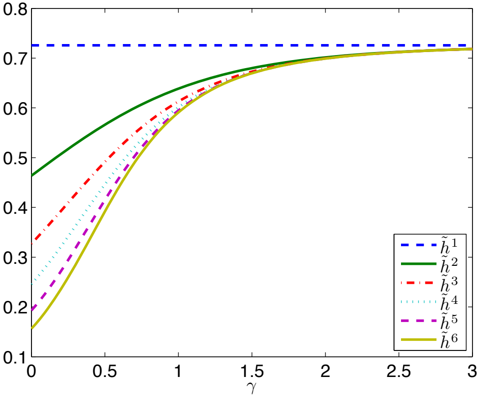
Numerical verification of the condition of the theorem.

Ansatz without Alternating Signs

For the ansatz

$$U_p(\zeta) = A\,|p|^{-\gamma}\, e^{-\frac{p^2}{3}\zeta^2}, \quad p \in \mathbb{Z}_{\rm odd},$$

no extrema points occur in the reduced energy dependence on γ.

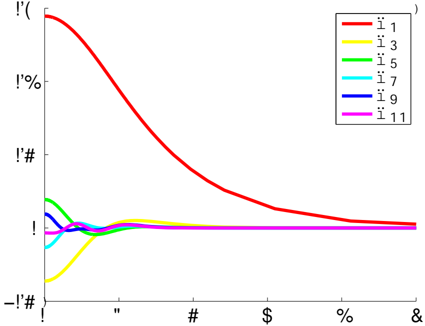


Direct Numerical Solution of Truncated NLS System

NLS System

$$U_p''(\zeta) - p^2 U_p + \frac{2}{3}p^2 \left(3U_p \sum |U_q|^2 + \sum U_q U_r U_{p-q-r}\right) = 0$$

For up to 12 modes, the structure of sign alternations persists:



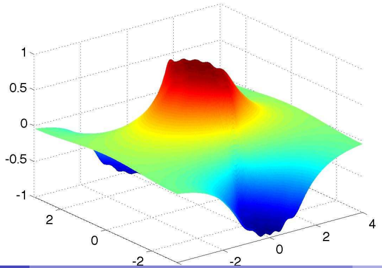
Equivalent integro-differential equation

Elliptic equation

$$(\partial_\zeta^2 + \partial_\phi^2)U = \frac{2}{3}\partial_\phi^2 \left[U^3 + 3\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} |U(\zeta,\theta)|^2 d\theta\right) U\right],$$

where

$$U(\theta,\zeta) = \sum_{p\in\mathbb{Z}_{\rm odd}} U_p(\zeta) e^{ip\theta}.$$

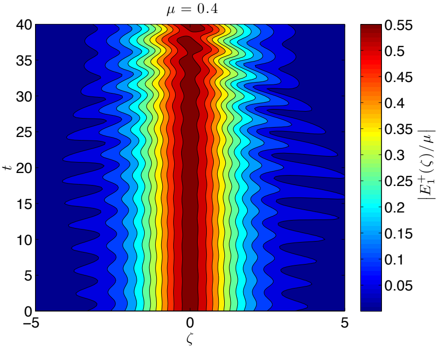


Time evolution of NLS Solitons in xNLCMEs

$$\partial_T E_p^+ + \partial_Z E_p^+ = \mathrm{i}pN_{2p}E_p^- + \frac{\mathrm{i}p}{3}\left[\sum E_q^+ E_r^+ E_{p-q-r}^+ + 3\left(\sum |E_q^-|^2\right) E_p^+\right]$$

$$\partial_T E_p^- - \partial_Z E_p^- = \mathrm{i}p\bar{N}_{2p}E_p^+ + \frac{\mathrm{i}p}{3}\left[\sum E_q^- E_r^- E_{p-q-r}^- + 3\left(\sum |E_q^+|^2\right) E_p^-\right].$$

$$E_p^{\pm}(Z,0) = \pm\mu U_p(\mu Z),\ p = 1$$

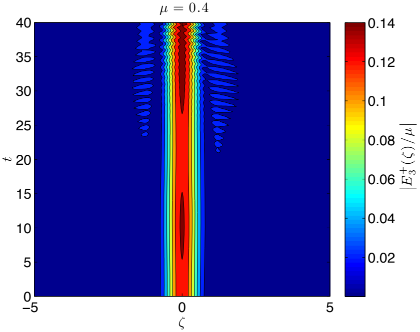


Time evolution of NLS Solitons in xNLCMEs

$$\partial_T E_p^+ + \partial_Z E_p^+ = \mathrm{i} p N_{2p} E_p^- + \frac{\mathrm{i}p}{3}\left[\sum E_q^+ E_r^+ E_{p-q-r}^+ + 3\left(\sum |E_q^-|^2\right) E_p^+\right]$$

$$\partial_T E_p^- - \partial_Z E_p^- = \mathrm{i} p \bar{N}_{2p} E_p^+ + \frac{\mathrm{i}p}{3}\left[\sum E_q^- E_r^- E_{p-q-r}^- + 3\left(\sum |E_q^+|^2\right) E_p^-\right].$$

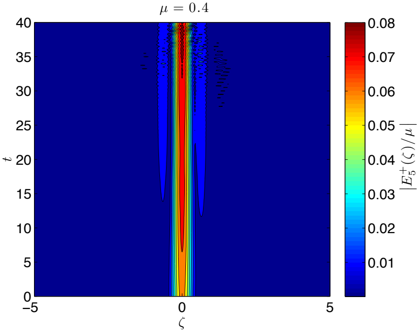
$$p = 3$$



Time evolution of NLS Solitons in xNLCMEs

$$\partial_T E_p^+ + \partial_Z E_p^+ = \mathrm{i} p N_{2p} E_p^- + \frac{\mathrm{i}p}{3}\left[\sum E_q^+ E_r^+ E_{p-q-r}^+ + 3\left(\sum |E_q^-|^2\right) E_p^+\right]$$
$$\partial_T E_p^- - \partial_Z E_p^- = \mathrm{i} p \bar{N}_{2p} E_p^+ + \frac{\mathrm{i}p}{3}\left[\sum E_q^- E_r^- E_{p-q-r}^- + 3\left(\sum |E_q^+|^2\right) E_p^-\right].$$

$$p = 5$$



Conclusion

Summary:

Our results suggest that the localized states are robust for the nonlinear periodic Maxwell model. Existence of such states do not eliminate a possibility of shocks for large amplitudes.

Further directions

- Prove the existence of localized solutions in the coupled NLS equations (or in the equivalent elliptic problem)
- Justify the coupled NLS equations in the original Maxwell system with periodic Dirac delta-distributions
- Consider localized solutions in the Maxwell system with bounded refractive index.

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