# Localized Travelling Waves in Nonlinear Schrödinger Lattices

Dmitry Pelinovsky, McMaster University, Canada
Thomas Melvin, Alan Champneys, University of Bristol, U.K.

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# Model

- Discrete nonlinear Schrödinger equation (DNLS) in 1-D

$$i\dot{u}_n(t) + \frac{u_{n+1}(t) - 2u_n(t) + u_{n-1}(t)}{h^2} + f(u_{n+1}, u_n, u_{n-1}) = 0.$$

- General nonlinear term $f$:
  - Cubic DNLS, $f = |u_n|^2 u_n$.
  - Ablowitz-Ladik $f = |u_n|^2(u_{n+1} + u_{n-1})$.
  - Salerno model

$$f = 2\alpha|u_n|^2 u_n + (1-\alpha)|u_n|^2(u_{n+1} + u_{n-1}).$$

  - Translationally invariant model

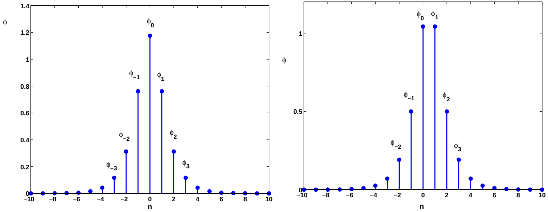
$$\begin{aligned} f &= \alpha_1|u_n|^2 u_n + \alpha_2|u_n|^2(u_{n+1} + u_{n-1}) + \alpha_3 u_n^2(\bar{u}_{n+1} + \bar{u}_{n-1}) \\ & \quad ... + \alpha_{10}(|u_{n+1}|^2 u_{n-1} + |u_{n-1}|^2 u_{n+1}). \end{aligned}$$

# More on translationally invariant model

- Stationary solutions $u_n(t) = \phi_n e^{i\omega t}$ satisfy the second-order difference map

$$-\omega\phi_n + \frac{\phi_{n+1} - 2\phi_n + \phi_{n-1}}{h^2} + f(\phi_{n+1}, \phi_n, \phi_{n-1}) = 0.$$

- Two solutions: on-site and inter-site discrete solitons




- When $\alpha_1 = \alpha_4 + \alpha_6$, $\alpha_5 = \alpha_6$, $\alpha_7 = \alpha_4 - \alpha_6$ and $\alpha_{10} = \alpha_8 - \alpha_9$, the difference map admits a continuous family of localized solutions $\phi_n = \phi(n - s)$, where $s \in \mathbb{R}$ (D.P., Nonlinearity 19, 2695 (2006)).

# Traveling waves in lattices

- Discrete nonlinear Schrödinger equation

$$i\dot{u}_n(t) + \frac{u_{n+1}(t) - 2u_n(t) + u_{n-1}(t)}{h^2} + f(u_{n+1}, u_n, u_{n-1}) = 0.$$

- Moving into the travelling frame $z = hn - 2ct$ gives a differential advance-delay equation. If $u_n(t) = \phi(z)e^{i\omega t}$,

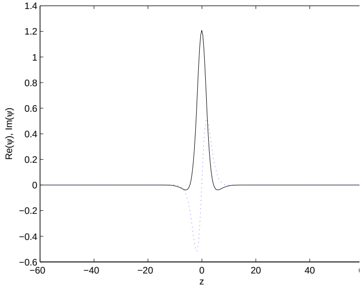
$$2ic\phi'(z) = \frac{\phi(z+h) - 2\phi(z) + \phi(z-h)}{h^2} - \omega\phi(z) + f(\phi(z+h), \phi(z)\phi(z-h)).$$

- Traveling waves satisfy the constraints:

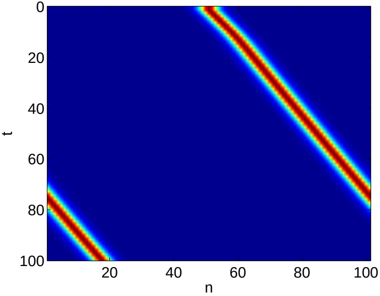
$$u_1(t) = u_0(t-\tau)e^{i\theta}, \quad u_2(t) = u_0(t-2\tau)e^{2i\theta}, \quad \text{etc.}$$

# Radiationless Solitons

- Localised solutions to a differential difference equation.
- Waves travel across a lattice without shedding *any* radiation.
- Homoclinic orbit to the zero state in a travelling frame.




(a)




(a)

# Difficulties

- In general, traveling wave solutions are weakly non-local.
- Eigenvalues on the imaginary axis in the linear spectrum give rise to radiation modes.
- Number of eigenvalues is finite for $c \neq 0$ but increases as $c \rightarrow 0$.
- In general there is at least one resonance.
- Amplitude of radiation modes are generally exponentially small in terms of a bifurcation parameter.

# Reformulation of existence problem

- Introduce parameters $\kappa \in \mathbb{R}_+$, $\beta \in [0, \pi]$

$$\omega = \frac{2}{h}\beta c + \frac{2}{h^2}(\cos(\beta)\cosh(\kappa) - 1),$$
$$c = \frac{1}{h\kappa}\sin(\beta)\sinh(\kappa),$$

- Scale out $h$ using $\phi(z) = \frac{1}{h}\Phi(Z)e^{i\beta Z}$, $Z = \frac{z}{h}$
- New differential advance-delay equation

$$i\sin(\beta)\left(2\frac{\sinh(\kappa)}{\kappa}\frac{d\Phi(Z)}{dZ} - \Phi(Z+1) + \Phi(Z-1)\right)$$
$$+\cos(\beta)\left(2\cosh(\kappa)\Phi(Z) - \Phi(Z+1) - \Phi(Z-1)\right)$$
$$-f(\Phi(Z+1)e^{i\beta}, \Phi(Z), \Phi(Z-1)e^{-i\beta}) = 0,$$

where $\kappa > 0$ and $\beta \in [0, \pi]$.

# Linear Spectrum

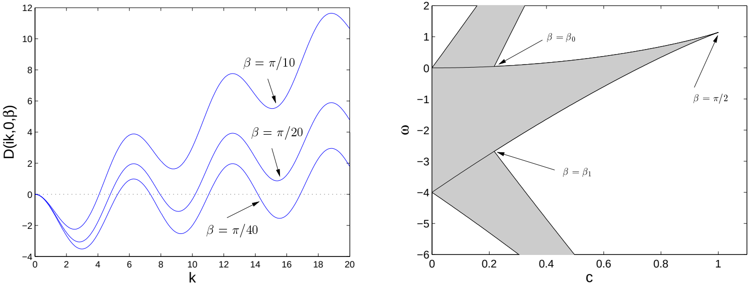
- Dispersion relation for the linear equation is obtained using $\Phi(Z) = e^{pZ}$

$$D(p;\kappa,\beta) \equiv 2\cos(\beta)(\cosh(p) - \cosh(\kappa)) + 2i\sin(\beta)\left(\sinh(p) - \frac{\sinh(\kappa)}{\kappa}p\right) = 0.$$

- there are finitely many imaginary roots $p = ik_n$, $n = 1, ..., m$ for any $\kappa > 0$ and $\beta \in (0, \pi)$
- if $\kappa = 0$, there exists a double root $k = 0$ of $D(ik; 0, \beta)$
- if $\kappa = 0$ and $\beta = \pi/2$, the zero root $k = 0$ is a triple root of $D(ik; 0, \beta)$
- if $\kappa = 0$ and $\beta \in \left(\beta_0, \frac{\pi}{2}\right)$ with $\beta_0 \approx \frac{\pi}{13}$, there exists only one imaginary root besides the double zero root.

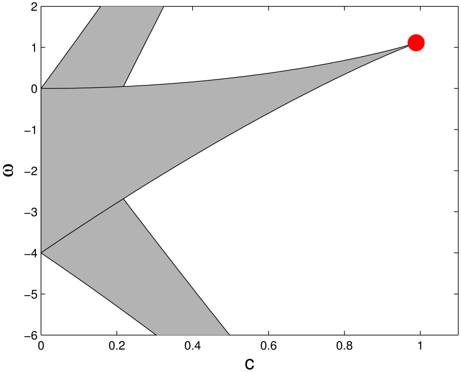
# Linear Spectrum

- Dispersion relation for the linear equation is obtained using $\Phi(Z) = e^{pZ}$



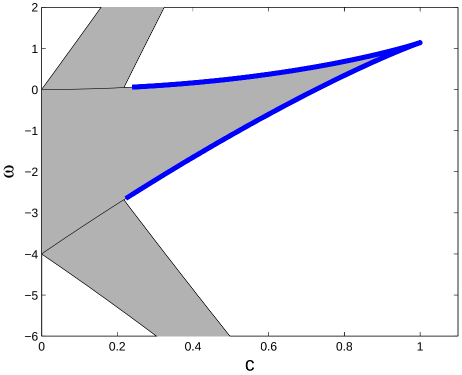
# Methods

- Normal forms and Melnikov integrals
  - Analysis of the normal form near $\kappa = 0$ and $\beta = \pi/2$ (D.P.,V.Rothos, Physica D 202, 16 (2005)).
  - Analysis of persistence of homoclinic orbits near the line $\kappa > 0$ and $\beta = \pi/2$ (D.P.,T.Melvin, A. Champneys, Physica D 236, 22 (2007)).



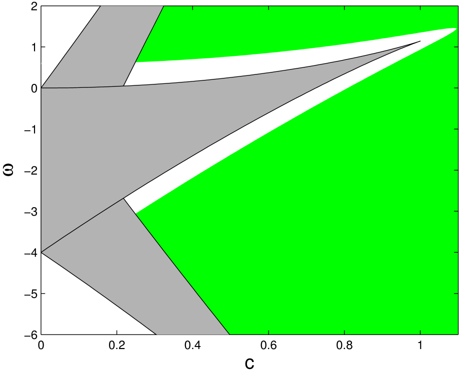
# Methods

- Stokes constant computation
  - Analysis of Stokes phenomena in a beyond all orders expansion for $\kappa = 0$ and $\beta \neq \pi/2$ (O. Oxtoby, I. Barashenkov, nlin/0610059 (2006)).



# Methods

- Pseudo-spectral decomposition
  - Numerical solutions of the differential advance-delay equation for $\kappa > 0$ and any $\beta$.



# Reduction of the differential advance-delay equation

- Write the main equation as

$$i\sin(\beta)\left(\Phi_+ - \Phi_- - 2\frac{\sinh(\kappa)}{\kappa}\Phi'(Z)\right) + \cos(\beta)\left(\Phi_+ + \Phi_- - 2\cosh(\kappa)\Phi\right) + f_r + if_i = 0,$$

where $\Phi_\pm = \Phi(Z \pm 1)$ and $f(\Phi_+ e^{i\beta}, \Phi, \Phi_- e^{-i\beta}) = f_r + if_i$.

- If $\beta = \frac{\pi}{2}$ and

$$\alpha_1 = 0, \quad \alpha_4 = \alpha_6, \quad \alpha_7 = 2\alpha_5,$$

the equation reduces to a scalar real-valued equation

$$2\frac{\sinh(\kappa)}{\kappa}\frac{d\Phi}{dZ} = \left[1 + (\alpha_2 - \alpha_3)\Phi^2 + \alpha_8(\Phi_+^2 + \Phi_+\Phi_- + \Phi_-^2) - (\alpha_9 + \alpha_{10})\Phi_+\Phi_-\right](\Phi_+ - \Phi_-).$$

# Assumption on existence of solutions

- Assumption: There exists a single-humped solution $\Phi_0(Z)$ for any $\kappa > 0$ and some parameters $(\alpha_2^{(0)}, \alpha_3^{(0)}, ...)$ s.t.

$$\Phi_0 \in H^1(\mathbb{R}): \quad \Phi_0(-Z) = \Phi_0(Z), \quad \lim_{|Z|\to\infty} e^{\kappa|Z|}\Phi_0(Z) = c_0.$$

- Any even solution is extended into a continuous family $\Phi_0(Z - s)$, $\forall s \in \mathbb{R}$.
- When $\alpha_8 = \alpha_9 = \alpha_{10} = 0$ and $\alpha_2 > \alpha_3$, the assumption is satisfied with the explicit solution

$$\Phi_0(Z) = \frac{\sinh\kappa}{\sqrt{\alpha_2 - \alpha_3}} \operatorname{sech}(\kappa Z), \quad \kappa > 0.$$

# Persistence of solutions I

Theorem: Under some assumptions on the linearized operator, the single-humped localized solution persists with respect to parameter continuations, such that $\|\Phi - \Phi_0\|_{H^1} \leq C\epsilon$, where $C > 0$ and $\epsilon = \max_j |\alpha_j - \alpha_j^{(0)}|$.

To the proof:

- Let $\Phi = \Phi_0 + U$, $\alpha_j = \alpha_j^{(0)} + \epsilon a_j$, and write the scalar equation as

$$L_+ U = N(U) + \epsilon F(\Phi_0 + U),$$

where $L_+$ is a differential advance-delay operator and

$$\|N(U)\|_{H^1} \leq C_1 \|U\|_{H^1}^2, \quad \|F(\Phi_0 + U)\|_{H^1} \leq C_2 \|\Phi_0 + U\|_{H^1}^3.$$

## To the proof:

- Notice that

$$L_+ : H^1_{\rm ev} \mapsto L^2_{\rm odd}, \quad N, F : H^1_{\rm ev} \mapsto H^1_{\rm odd}$$

  and

$$L_+ \Phi_0'(Z) = 0, \quad L_+ \frac{\partial \Phi_0}{\partial \kappa} = \frac{2(\kappa \cosh \kappa - \sinh \kappa)}{\kappa^2} \Phi_0'(Z).$$

- Assume that $L_+$ has no eigenvalues near $\mathrm{Re}(\lambda) = 0$ except for $\lambda = 0$ and that the zero eigenvalue is double. Then, invert $L_+$ on $L^2_{\rm odd}$ and use the Implicit Function Theorem.
- Although the continuous spectrum of $L_+$ extends on the imaginary axis $\mathrm{Re}(\lambda) = 0$, the entire spectrum is shifted off the imaginary axis in the exponentially weighted $H^1$ space.

# Persistence of solutions II

Theorem: Under additional assumptions on the linearized operator, the single-humped localized solution persists along the curve on $(\kappa, \beta)$-plane with respect to parameter continuations, such that $\|\Phi - \Phi_0\|_{H^1} \leq C(\epsilon + \mu)$ if and only if $\Delta(\epsilon, \mu) = 0$, where $\mu = \cot\beta$, $\alpha_j = \alpha_j^{(0)} + \epsilon a_j$ and $\Delta(\epsilon, \mu)$ is a Melnikov integral

$$\Delta(\epsilon, \mu) = \int_{\mathbb{R}} W_0(Z;0)[N_-(U,V) + F_-(\Phi_0 + U, V; \epsilon, \mu)]dZ,$$

where

- $W_0$ is an eigenfunction of the adjoint operator for the zero eigenvalue,
- $N_-$ is the unperturbed vector field with quadratic and cubic terms in $\Phi(Z) - \Phi_0(Z) = U(Z) + iV(Z)$, and
- $F_-$ contain linear and nonlinear terms in $\Phi_0 + U$ and $V$ related to the perturbations in $\mu = \cot\beta$ and $\epsilon$.

It is clear that $\Delta(0,0) = 0$.

# Example I : Salerno model

The model is

$$i\dot{u}_n(t) + \frac{u_{n+1}(t) - 2u_n(t) + u_{n-1}(t)}{h^2}$$
$$+2\alpha|u_n|^2 u_n + (1-\alpha)|u_n|^2(u_{n+1} + u_{n-1}) = 0.$$

- If $\alpha = 0$, the family of solutions with $\beta = \frac{\pi}{2}$ is a part of a two-parameter family. $\Longrightarrow \Delta(0,\mu) = 0$ for any $\mu \in \mathbb{R}$.
- If $\Delta(\epsilon, 0) \neq 0$ for $\epsilon \neq 0$, the family can not be continued in $\epsilon$.
- Explicit computation shows that

$$\partial_\epsilon \Delta(0,0) = \int_{\mathbb{R}} W_0(Z;0)\Phi_0^3(Z)dZ \approx -\frac{\kappa^2}{2}\int_{\mathbb{R}} \frac{d\zeta}{\cosh^3\zeta} < 0,$$

  for small $\kappa > 0$.
- Therefore, the family of exact solutions of the AL lattice does not persist in the Salerno model near $\beta = \frac{\pi}{2}$.

# Example II : Translationally invariant model

The model is

$$\begin{aligned} i\dot{u}_n(t) &+ \frac{u_{n+1}(t) - 2u_n(t) + u_{n-1}(t)}{h^2} \\ &+ \alpha_1 |u_n|^2 u_n + \alpha_2 |u_n|^2 (u_{n+1} + u_{n-1}) + \alpha_3 u_n^2 (\bar{u}_{n+1} + \bar{u}_{n-1}) \\ &\dots + \alpha_{10}(|u_{n+1}|^2 u_{n-1} + |u_{n-1}|^2 u_{n+1}) = 0. \end{aligned}$$

The exact solution exists for $\alpha_1 = \alpha_4 = \dots = 0$ and $\alpha_2 > \alpha_3$.

- If $\partial_\mu \Delta(0,0) \neq 0$ for any $\kappa > 0$, there exists a unique continuation of the solution $\Phi_0$ near the line $\beta = \frac{\pi}{2}$.
- Explicit computation shows that

$$\begin{aligned} \partial_\mu \Delta(0,0) &= 2\alpha_3 \int_{\mathbb{R}} W_0(Z;0)\Phi^2(\Phi_+ + \Phi_-)dZ \\ &\approx \frac{4\kappa^2 \alpha_3}{(\alpha_2 - \alpha_3)^{3/2}} \int_{\mathbb{R}} \left(1 - 2\text{sech}^2\zeta\right) \text{sech}^3\zeta d\zeta \neq 0, \end{aligned}$$

for small $\kappa > 0$.

# Example II : Translationally invariant model

In addition,

$$\begin{aligned}\partial_\epsilon \Delta(0,0) &= \int_{\mathbb{R}} W_0(Z;0)\Big[\alpha_1\Phi^3 + (\alpha_4-\alpha_6)\Phi(\Phi_+^2+\Phi_-^2) \\ &\quad +(\alpha_7-2\alpha_5)\Phi\Phi_+\Phi_-\Big]\,dZ\,,\end{aligned}$$

which is zero for $\alpha_1 = 0$, $\alpha_4 = \alpha_6$, and $\alpha_7 = 2\alpha_5$.

The localized solution persists on the line $\beta = \frac{\pi}{2}$ if

$$\alpha_1 = 0, \quad \alpha_4 = \alpha_6, \quad \alpha_7 = 2\alpha_5.$$

# Pseudo-Spectral method

- Use a pseudo-spectral method to transform differential advance-delay equation $\rightarrow$ system of algebraic equations

$$\Phi(Z_i) = \sum_{j=1}^{N} a_j \cos\left(\frac{2\pi j}{L} Z_i\right) + i b_j \sin\left(\frac{2\pi j}{L} Z_i\right).$$

- Solutions are defined on a large finite domain $L$ at the collocation points $Z_i = \frac{Li}{2(N+1)}$.
- Solutions have generally a non-zero radiation tail near the end points $Z = \pm L/2$. To measure the tail, we use the signed amplitude

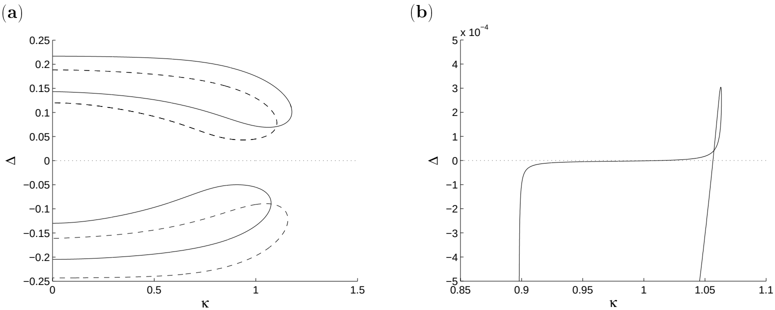
$$\Delta = \mathrm{Im}(\Phi(L/2)).$$

# Example I : Salerno model

The model is

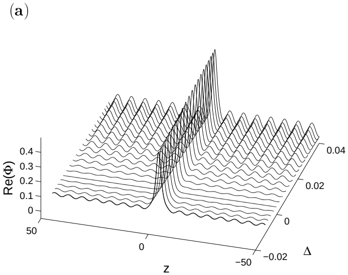
$$i\dot{u}_n(t) + \frac{u_{n+1}(t) - 2u_n(t) + u_{n-1}(t)}{h^2} + 2\alpha|u_n|^2 u_n + (1-\alpha)|u_n|^2(u_{n+1} + u_{n-1}) = 0.$$

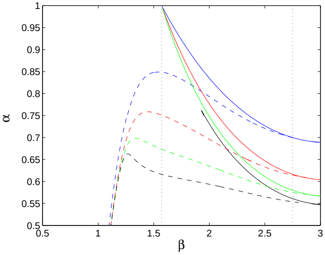
Localised solutions do not exist for $\alpha = 0.9, 1.1$, $\beta = 0.35\pi, 0.65\pi$ (left) but do exist for $\alpha = 0.7$, $\beta = 0.875\pi$ (right).



# Example I : Salerno model

Profiles of solutions for real part of $\Phi(Z)$ versus tail amplitude $\Delta$ (left). Solution branches for a fixed $\kappa > 0$: one-humped for $\beta > \frac{\pi}{2}$ and two-humped for $\beta < \pi/2$ (right).



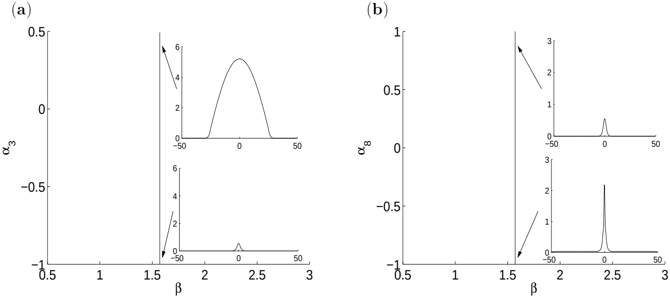



# Example II : Translationally invariant model

The model is

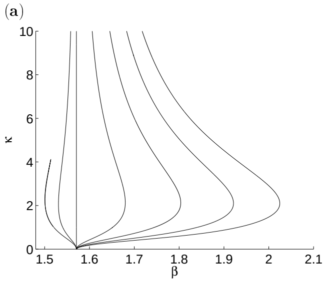
$$
\begin{aligned}
i\dot{u}_n(t) \quad &+ \quad \frac{u_{n+1}(t) - 2u_n(t) + u_{n-1}(t)}{h^2} \\
&+ \quad \alpha_1 |u_n|^2 u_n + \alpha_2 |u_n|^2 (u_{n+1} + u_{n-1}) + \alpha_3 u_n^2 (\bar{u}_{n+1} + \bar{u}_{n-1}) \\
&\quad \ldots + \alpha_{10} (|u_{n+1}|^2 u_{n-1} + |u_{n-1}|^2 u_{n+1}) = 0.
\end{aligned}
$$

If $\alpha_1 = 0$, $\alpha_4 = \alpha_6$, $\alpha_7 = 2\alpha_5$, the solution persists for $\beta = \frac{\pi}{2}$.



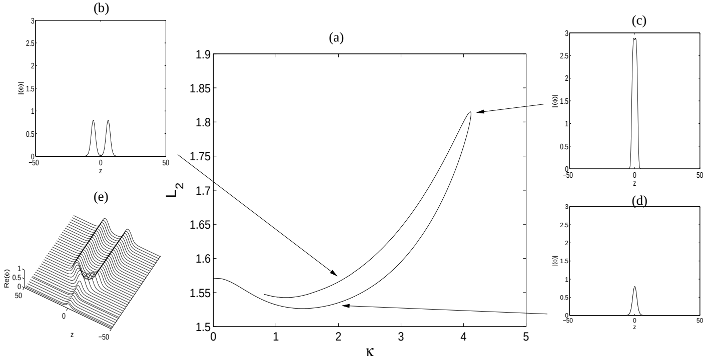
# Example II : Translationally invariant model

The solution persists generally as a one-parameter curve on the parameter plane



# Example II : Translationally invariant model

Branches of single-humped solutions connect to branches of double-humped solutions.



# Conclusion

- Traveling localized waves are still generic in many discrete NLS equations in spite of the presence of resonances.
- One-parameter curves in non-integrable lattices are more structurally stable with respect to perturbations than two-parameter curves in near-integrable lattices.
- Traveling localized waves in the translationally invariant model are stable with respect to time-dependent perturbations.
- Salerno model also has traveling localized wave solutions (away from the integrable Ablowitz–Ladik limit).