Nonlinearity management in time-periodic NLS systems

Dmitry Pelinovsky (McMaster University, Canada)

Joint work with

Vadim Zharnitsky (University of Illinois at Urbana-Champaign)

Panos Kevrekidis (University of Massachusetts)

References:

Phys. Rev. Lett. 91, 240201 (2003)

Chaos 15, 037105 (2005)

J. Phys. A: Math. Gen. 39, 479 (2006)

J. Diff. Eqs. 220, 85 (2006)

Background and motivations

Time-periodic NLS equation

$$iu_t = -\Delta u + \gamma(t)|u|^2 u + V(x)u,$$

where

- $u(x,t) : \mathbb{R}^d \times \mathbb{R}_+ \mapsto \mathbb{C}$ is a classical solution
- $\gamma(t + t_0) = \gamma(t)$ is a periodic coefficient
- $V(x) \geq 0$ is a (decaying, parabolic, or periodic) potential

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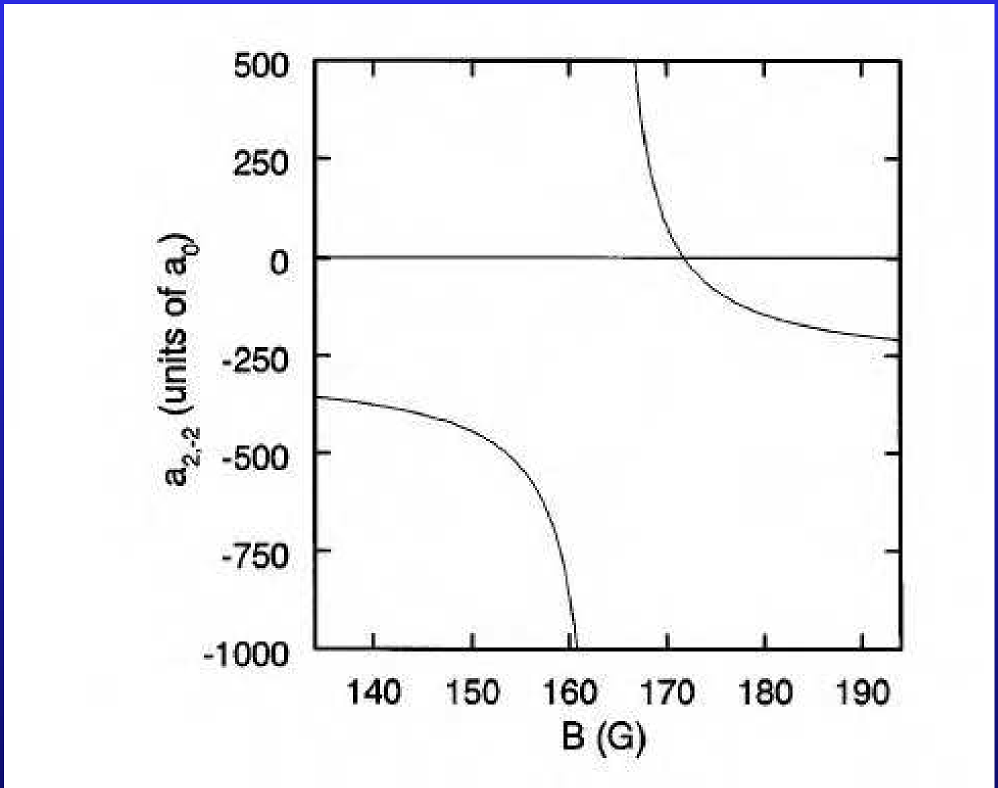
Applications:

- Feshbach resonance in Bose-Einstein condensates
- optical pulse propagation in layered optical media

Physical experiments (1998)

Scattering length versus magnetic field



Feshbach resonance in ^{85}Rb

Mathematical problems

Time-periodic NLS equation

$$iu_t = -\Delta u + \gamma(t)|u|^2 u + V(x)u,$$

- homogenization in the limit of short and large-amplitude variations of $\gamma(t)$
  $\Rightarrow$ derivation of the averaged NLS equation

Mathematical problems

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  $\Rightarrow$ derivation of the averaged NLS equation
- arrest of blowup of large-norm solutions in dimensions $d \geq 2$
  $\Rightarrow$ proof of global well-posedness of the averaged equation

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- homogenization in the limit of short and large-amplitude variations of $\gamma(t)$
  $\Rightarrow$ derivation of the averaged NLS equation
- arrest of blowup of large-norm solutions in dimensions $d \geq 2$
  $\Rightarrow$ proof of global well-posedness of the averaged equation
- radiative decay of localized solutions supported by $V(x)$
  $\Rightarrow$ decay law of the amplitude of localized solutions

Averaging theory

Time-periodic NLS equation

$$iu_t = -u_{xx} + \gamma_0 |u|^2 u + \frac{1}{\epsilon}\gamma\left(\frac{t}{\epsilon}\right)|u|^2 u,$$

where

- $V(x) \equiv 0$ for simplicity
- $d = 1$ without loss of generality
- $\epsilon \to 0$ is the limit of short and large-amplitude variations of $\gamma(t)$, such that

$$\gamma(\tau + 1) = \gamma(\tau), \qquad \int_0^1 \gamma(\tau) d\tau = 0.$$

Formal transformation

Local transformation

$$u(x,t) = v(x,t)\ \exp\left(-i\gamma_{-1}(\tau)|v|^2(x,t)\right)$$

where $\gamma_{-1}(\tau)$ is the mean-zero antiderivative of $\gamma(\tau)$.

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Equivalent NLS equation

$$iv_t = -v_{xx} + \gamma_0|v|^2 v + 2i\gamma_{-1}(\tau)\left(v^2\bar{v}_{xx} + 2|v_x|^2 v + v_x^2\bar{v}\right)$$
$$-\gamma_{-1}^2(\tau)\left(\left(|v|_x^2\right)^2 + 2|v|_{xx}^2|v|^2\right)v.$$

Note: one-step averaging procedure after the transformation

Review of averaging procedures

Methods of solution:

- canonical transformations of the Hamiltonian
- asymptotic multiscale expansions
- rigorous estimation of the error bounds

References in dispersion management:

- V. Zharnitsky, E. Grenier, S. Turitsyn, C. Jones, Physica D 152, 794 (2001)
- D.Pelinovsky, V. Zharnitsky, SIAM J. Appl. Math. 63, 745 (2003)

Asymptotic multi-scale expansions

Asymptotic expansion

$$v(x,t) = w(x,t) + \epsilon v_1(x,t,\tau) + \mathrm{O}(\epsilon^2)$$

where τ is fast time and t is slow time.

Asymptotic multi-scale expansions

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The averaged NLS equation

$$iw_t = -w_{xx} + \gamma_0 |w|^2 w - \sigma^2 \left(\left(|w|_x^2\right)^2 + 2|w|_{xx}^2 |w|^2 \right) w,$$

where σ^2 is the mean value of $\gamma_{-1}^2(\tau)$

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The first-order correction

$$v_1 = 2(\gamma_{-1})_{-1} \left(w^2 \bar{w}_{xx} + 2|w_x|^2 w + w_x^2 \bar{w} \right)$$
$$-i(\gamma_{-1}^2 - \sigma^2)_{-1} \left(\left(|w|_x^2\right)^2 + 2|w|_{xx}^2 |w|^2 \right) w,$$

Properties of averaged NLS equation

Hamiltonian form of the time-periodic NLS equation

$$H = \int_{\mathbb{R}} \left(|u_x|^2 + \frac{1}{2}\gamma_0 |u|^4 + \frac{1}{2\epsilon}\gamma\left(\frac{t}{\epsilon}\right)|u|^4 \right) dx.$$

Hamiltonian form of the averaged NLS equation

$$H = \int_{\mathbb{R}} \left(|w_x|^2 + \frac{1}{2}\gamma_0 |w|^4 + \sigma^2 |w|^2 \left(|w|^2_x\right)^2 \right) dx.$$

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Hamiltonian form of the averaged NLS equation

$$H = \int_{\mathbb{R}} \left(|w_x|^2 + \frac{1}{2}\gamma_0 |w|^4 + \sigma^2 |w|^2 \left(|w|_x^2\right)^2 \right) dx.$$

Local well-posedness in $H^\infty = \cap_{n\geq 0} H^n(\mathbb{R})$:

Let $w(x,0) \in H^\infty$. There exists $T > 0$ such that the averaged NLS equation possess a unique solution $w(x,t) \in C^1([0,T], H^\infty)$.

M. Poppenberg, Nonlinear Anal. Theory 45, 723 (2001)

Obstacle in the nonlocal method

Nonlocal transformation

$$u = v(x,t)\ \exp\left(-i\phi(x,t)\right), \quad \phi = \frac{1}{\epsilon}\int_0^t \gamma\left(\frac{t'}{\epsilon}\right)|v|^2(x,t')dt'$$

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Equivalent NLS equation

$$iv_t = -v_{xx} + \gamma_0|v|^2 v + 2i\phi_x v_x + i\phi_{xx} v + (\phi_x)^2 v$$

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Equivalent NLS equation

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Formal averaging procedure produces an averaged NLS equation

$$iw_t = -w_{xx} + \gamma_0|w|^2 w + i\nu_1(2|w|_x^2 w_x + |w|_{xx}^2 w) + (\nu_1^2 + \sigma^2)(|w|_x^2)^2 w$$

where ν_1 is the average of the anti-derivative of $\gamma(\tau)$.

Failure of the nonlocal method

The first-order correction

$$v_1 = -(\gamma_{-1})_{-1}\left(2|w|_x^2 w_x + |w|_{xx}^2 w\right) - i(\gamma_{-1}^2 - \sigma^2)_{-1}\left(|w|_x^2\right)^2 w,$$

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The non-zero average term:

$$\int_0^1 \gamma(\tau)\left(\bar{w}(x,t)v_1(x,\tau,t) + w(x,t)\bar{v}_1(x,\tau,t)\right)d\tau$$
$$= 2\sigma^2\left(|w|^2|w|_x^2\right)_x \neq 0.$$

Failure of the nonlocal method

The first-order correction

$$v_1 = -(\gamma_{-1})_{-1}\left(2|w|^2_x w_x + |w|^2_{xx} w\right) - i(\gamma^2_{-1} - \sigma^2)_{-1}\left(|w|^2_x\right)^2 w,$$

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$$= 2\sigma^2\left(|w|^2|w|^2_x\right)_x \neq 0.$$

The first-order correction ϕ_1 grows linearly in τ and the second-order correction term v_2 grows quadratically in τ.
$\Rightarrow$ The averaging procedure fails for nonlocal (integral) equations.

Resolving the problem

The extended system of local equations

$$
\begin{aligned}
iv_t &= -v_{xx} + \gamma_0 |v|^2 v + 2i\phi_x v_x + i\phi_{xx} v + (\phi_x)^2 v \\
\phi_t &= \frac{1}{\epsilon}\gamma\left(\frac{t}{\epsilon}\right)|v|^2
\end{aligned}
$$

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Asymptotic expansions for v and ϕ result in the system

$$\begin{aligned} iw_t &= -w_{xx} + \gamma_0 |w|^2 w - 2i\varphi_x w_x - i\varphi_{xx} w + (\varphi_x)^2 w + \sigma^2 (|w|_x^2)^2 w \\ \varphi_t &= 2\sigma^2 \left(|w|^2 |w|_x^2\right)_x . \end{aligned}$$

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The function $\tilde{w} = w(x,t)e^{-i\varphi(x,t)}$ solves the *correct* averaged NLS equation.

Nonlinear bound states

ODE reductions for nonlinear bound states

$$w(x,t) = \Phi(x) e^{i\omega t}, \qquad \left(\frac{d\Phi}{dx}\right)^2 = \frac{(2\omega + \gamma_0 \Phi^2)}{2(1 + 4\sigma^2\Phi^4)}\Phi^2$$

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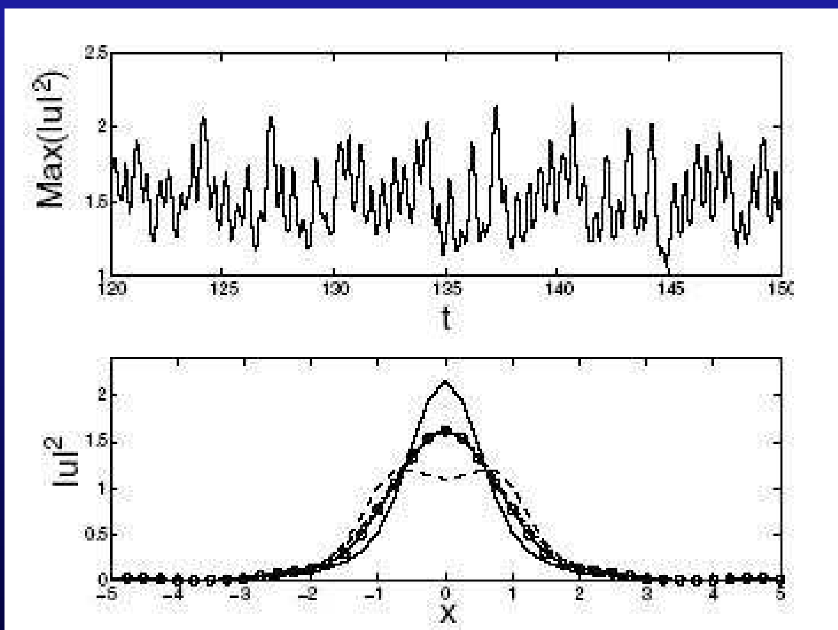
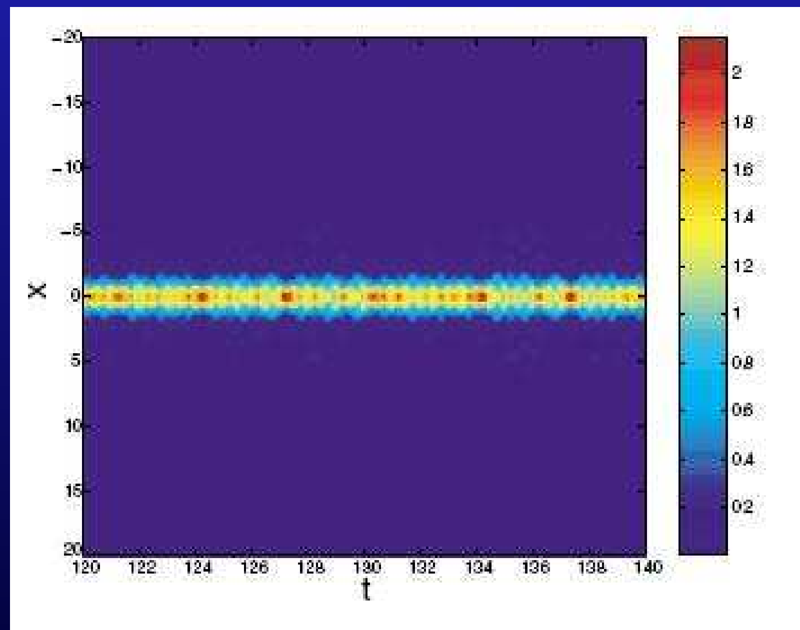
Temporal evolution of the bound state in the parabolic potential $V(x) \sim x^2$





Arrest of blowup

Hamiltonian of the averaged NLS equation

$$H = H_1(w) + \gamma_0 H_2(w),$$

where $d \geq 2$, $\gamma_0 < 0$, and

$$H_1 = \int_{\mathbb{R}^d} \left(|\nabla w|^2 + \sigma^2 |w|^2 \left(\nabla |w|^2 \right)^2 \right) dx, \qquad H_2 = \frac{1}{2} \int_{\mathbb{R}^d} |w|^4 dx.$$

- Blowup occurs at $\sigma = 0$ (no nonlinearity management)
- Blowup may occur in the time-periodic NLS equation
  V. Konotop and P. Pacciani, Phys. Rev. Lett. 94, 240405 (2005)
- We show that blowup never occurs in the averaged NLS equation with $\sigma \neq 0$ (strong nonlinearity management)

Local and global solutions

- Local solutions of the averaged NLS equation in $H^\infty(\mathbb{R}^d)$
  C.E. Kenig, *The Cauchy problem for the Quasilinear Schroödinger Equation* (2002)

- Local solutions of the time-periodic NLS equation in $H^1(\mathbb{R}^d)$
  T. Cazenave, *Semilinear Schrödinger equations* (2003)

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- Difficulty: no local existence of the averaged NLS equation is proved in $H^1(\mathbb{R})$
- Assuming the local existence for the averaged NLS equation in $H^1(\mathbb{R}^d)$, we show that the solution remains globally in $H^1(\mathbb{R}^d)$, so that the standard blow-up mechanism for the focusing NLS equation with $d \geq 2$ does not occur.

Proof of arrest of blow-up

Gagliardo–Nirenberg inequality

$$\|f\|_{L^r} \le \|f\|_{L^p}^{\theta} \|f\|_{L^q}^{1-\theta},$$

where $1 \le p, q \le \infty$, $0 < \theta < 1$, and $r^{-1} = \theta p^{-1} + (1-\theta) q^{-1}$, results in the inequality

$$\|w\|_{L^4} \le \|w\|_{L^6}^{3/4} \|w\|_{L^2}^{1/4}.$$

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A modification of the Sobolev embedding theorem

$$\|f\|_{L^2} \leq C \left(\|\nabla f\|_{L^2} + \|f\|_{L^2} \right),$$

results in the inequality

$$\|w\|_{L^6}^6 \leq C \left(H_1(w) + \|w\|_{L^3}^6 \right)$$

Proof of arrest of blow-up

Again, the Gagliardo–Nirenberg inequality results in the inequality

$$\|w\|_{L^3} \le \|w\|_{L^6}^{1/2} \|w\|_{L^2}^{1/2},$$

such that for any $\mu > 0$, e.g. for $\mu < 1/(2C)$,

$$\|w\|_{L^6}^6 \le C \left(H_1(w) + \mu \|w\|_{L^6}^6 + \frac{1}{2\mu} \|w\|_{L^2}^6 \right).$$

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As a result,

$$\|w\|_{L^4}^4 \leq \mu H_1(w) + C \left(\|w\|_{L^2}^2 + \|w\|_{L^2}^4 \right)$$
$$H_1(w) \leq C \left(H(w) + P(w) + P^2(w) \right),$$

where $H(w)$ and $P(w)$ are constant in the time evolution.

Strong versus weak managements

Weak nonlinearity management

$$\frac{1}{\epsilon}\gamma\left(\frac{t}{\epsilon}\right) \mapsto \gamma\left(\frac{t}{\epsilon}\right)$$

reduces the averaged NLS equation to the form

$$iw_t = -\Delta w + \gamma_0 |w|^2 w - \epsilon^2 \sigma^2 \left(\left|\nabla |w|^2\right|^2 + 2|w|^2 \Delta |w|^2 \right) w,$$

where ϵ^2 is small.

T. Yang and W. Kath, Opt. Lett. 22, 985 (1997)
F. Abdullaev, J. Caputo, R. Kraenkel and B. Malomed, Phys. Rev. A 67, 013605 (2003)

Strong versus weak managements

Contradiction:

- No blow-up occurs in the averaged NLS equation for any $\epsilon^2\sigma^2 \neq 0$
- Blow-up *may* occur in the time-periodic NLS equation for small ϵ^2

Strong versus weak managements

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Simplifications:

- $d = 2$ (critical blow-up)
- exact ODE reduction by using the method of moments

$$\ddot{R}(t) = \frac{\alpha + \beta\gamma(t/\epsilon)}{R^3},$$

where $\alpha, \beta = \mathrm{O}(1)$ as $\epsilon \to 0$ and $\beta > 0$.

ODE analysis

ODE with $\beta > 0$:

$$\ddot{R}(t) = \frac{\alpha + \beta\gamma(t/\epsilon)}{R^3}.$$

Montesinos, Perez-Garcia, Torres, Physica D 191, 193 (2004)

- Sufficient condition for blow-up

$$\alpha < -\beta \max_{0\leq\tau\leq 1}(\gamma).$$

- Necessary condition for bounded oscillations

$$\alpha > -\beta \max_{0\leq\tau\leq 1}(\gamma).$$

Strong versus weak managements

Contradiction:

- Strong management $\beta \gg |\alpha|$ results in the blow-up arrest
- Weak management $\beta \sim |\alpha|$ *may* result in blow-up

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Consider the averaging method for $\gamma = \sin(2\pi\tau)$:

$$R = r(t) + \epsilon^2 R_2(\tau, r) + \epsilon^4 R_4(\tau, r) + \mathrm{O}(\epsilon^6),$$

where the mean-value term $r(t)$ satisfies the averaged equation

$$\ddot{r} = \frac{\alpha}{r^3} + \epsilon^2 \frac{3\beta^2}{2r^7} + \epsilon^4 \frac{15\alpha\beta^2}{2r^{11}} + \mathrm{O}(\epsilon^6),$$

where $\alpha < 0$ and $\beta > 0$

Failure of the averaged equation

Effective potential

$$U(r) = \frac{\alpha}{2r^2} + \epsilon^2 \frac{\beta^2}{4r^6} + \epsilon^4 \frac{3\alpha\beta^2}{4r^{10}} + \mathrm{O}(\epsilon^6)$$

with $\alpha < 0$ and $\beta > 0$.

- $\epsilon = 0$: blow-up in a finite time
- $\mathrm{O}(\epsilon^2)$: blow-up is arrested
- $\mathrm{O}(\epsilon^4)$: blow-up may occur depending on the ratio between parameters α and β

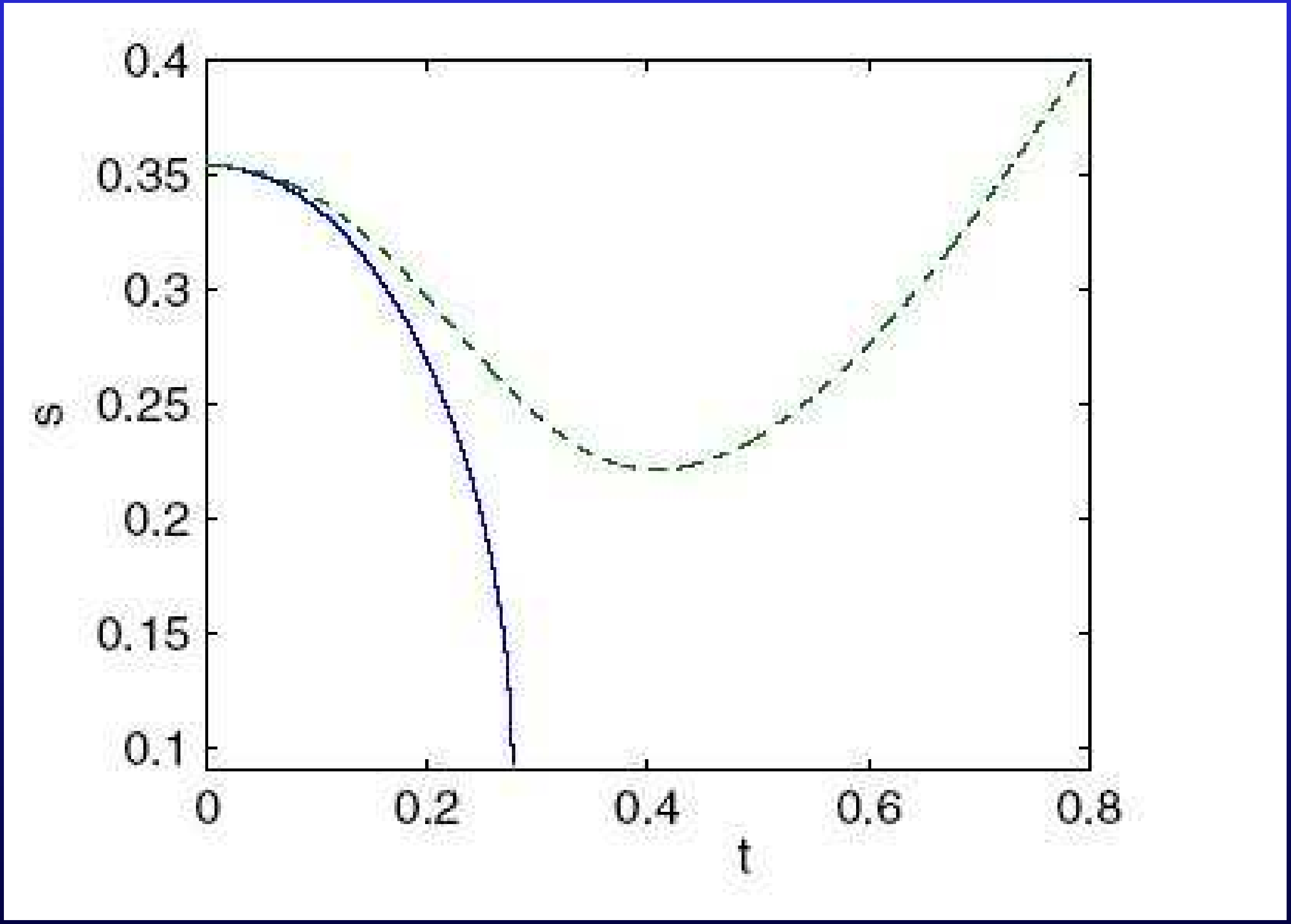
The exact threshold $\alpha = -\beta$ can not be found from the truncated averaged equation!

Numerical computations

$\alpha = -20$, $\beta = 8$

Solid - time-periodic NLS equation

Dashed - averaged NLS equation



Other and open problems

- Local well-posedness of the averaged NLS equation in $H^1(\mathbb{R})$
- Error bounds on the distance between time-periodic and averaged NLS equations
- Decay rate on radiative damping of localized solutions in the time-periodic NLS equation
- Sharp bounds on the initial data for blow-up in the time-periodic NLS equation
- Modeling of Feshbach resonance in different potentials (such as the periodic potential $V(x)$) and gap solitons