Wave breaking in the Ostrovsky–Hunter equation

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AMS, Notre Dame, 7 November 2010

References:

Yu. Liu, D.P., A. Sakovich, SIAM J. Math. Anal. 42, 1967-1985 (2010)
Yu. Liu, D.P., A. Sakovich, Dynamics of PDE 6, 291-310 (2009)
D.P., A. Sakovich, Communications in PDE 35, 613-629 (2010)

Ostrovsky equation for rotating fluid

The **Ostrovsky equation** is a model for small-amplitude long waves in a rotating fluid of a finite depth [Ostrovsky, 1978]:

$$(u_t + uu_x - \beta u_{xxx})_x = \gamma u,$$

where β and γ are real coefficients.

When $\beta = 0$ and $\gamma = 1$, the Ostrovsky equation is

$$(u_t + uu_x)_x = u,$$

and is known under the names of

- the short-wave equation [Hunter, 1990];
- Ostrovsky–Hunter equation [Boyd, 2005];
- reduced Ostrovsky equation [Stepanyants, 2006];
- the Vakhnenko equation [Vakhnenko & Parkes, 2002].

Short-pulse equation

The **short-pulse equation** is a model for propagation of ultra-short pulses with few cycles on the pulse scale [Schäfer, Wayne 2004]:

$$u_{xt} = u + \frac{1}{6}\left(u^3\right)_{xx},$$

where all coefficients are normalized thanks to the scaling invariance.

The short-pulse equation

- replaces the nonlinear Schrödinger equation for short wave packets
- features exact solutions for modulated pulses
- enjoys inverse scattering and an infinite set of conserved quantities

Background of our work

- A. Stefanov (2010) considered a family of the generalized short-pulse equations
$$u_{xt} = u + (u^p)_{xx}$$
and proved global existence and scattering to zero for *small* initial data if $p \geq 4$.

- We (2010) proved both global well-posedness for *small* initial data and wave breaking for *large* initial data if $p = 3$.

- We (2010) proved wave breaking for sufficiently *large* initial data if $p = 2$ but found no proof of global existence for *small* initial data.

- C. Holliman & A. Himonas (2010) proved the lack of uniform continuity with respect to initial data for the Hunter-Saxton equation
$$(u_t + uu_x)_x = (u_x)^2.$$

Wave breaking results

The inviscid Burgers equation $u_t + uu_x = 0$ develops wave breaking in a finite time for any initial data $u(0,x) = u_0(x)$ if $u_0(x) \in C^1$ and there is a point x_0 such that $u_0'(x_0) < 0$. In other words, there exists a finite time $T \in (0,\infty)$ such that

$$\liminf_{t \uparrow T} \inf_x u_x(t,x) = -\infty, \quad \text{while} \quad \limsup_{t \uparrow T} \sup_x |u(t,x)| < \infty.$$

Moreover, the blow-up time is computed by the method of characteristics:

$$T = \inf_\xi \left\{ \frac{1}{|u_0'(\xi)|} : \quad u_0'(\xi) < 0 \right\}.$$

Regarding the reduced Ostrovsky equation $p = 2$, it was found that

Theorem (Hunter, 1990)

Let $u_0(x) \in C^1(\mathbb{S})$, where $\mathbb{S}$ is a circle of unit length, and define

$$\inf_{x \in \mathbb{S}} u_0'(x) = -m \quad \text{and} \quad \sup_{x \in \mathbb{S}} |u_0(x)| = M.$$

If $m^3 > 4M(4+m)$, a smooth solution $u(t,x)$ breaks down at a finite time.

Our work

Our goal is to find several sufficient conditions for finite-time blow-up in the reduced Ostrovsky equation and to compare their sharpness using numerical simulations.

There exist infinitely many conserved quantities of the Ostrovsky-Hunter equations, which are not useful:

$$
\begin{aligned}
E_0 &= \int_{\mathbb{R}} u^2 dx, \\
E_{-1} &= \int_{\mathbb{R}} \left(\frac{1}{3} u^3 + (\partial_x^{-1} u)^2 \right) dx, \\
\dots
\end{aligned}
$$

Other nonlinear evolution equations are known to exhibit integrability and finite-time blow-up in a similar context: the Camassa-Holm equation, the Dagesperis–Processi equation, and their multi-component generalizations.

Local well-posedness

Cauchy problem on a circle $\mathbb{S}$ of unit length:

$$\begin{cases} u_t + uu_x = \partial_x^{-1} u, & t > 0, \\ u(0,x) = u_0(x), \end{cases}$$

where

$$\partial_x^{-1} u := \int_0^x u(t,x')dx' - \int_{\mathbb{S}} \int_0^x u(t,x')dx'dx.$$

Lemma

Assume that $u_0(x) \in H^s(\mathbb{S})$, $s > \frac{3}{2}$ and $\int_{\mathbb{S}} u_0(x)\, dx = 0$. Then there exist a maximal time $T = T(u_0) > 0$ and a unique solution $u(t,x)$ to the Cauchy problem such that

$$u(t,x) \in C([0,T); H^s(\mathbb{S})) \cap C^1([0,T); H^{s-1}(\mathbb{S})).$$

Moreover, the solution depends continuously on the initial data.

- Proofs back to Schäfer & Wayne (2004) and Stefanov et al. (2010).
- The assumption $\int_{\mathbb{S}} u_0(x)\, dx = 0$ on the initial data u_0 is necessary.

Finite-time wave breaking

- The maximal time $T > 0$ is independent of $s > \frac{3}{2}$.
- Let $u_0(x) \in H^s(\mathbb{S})$, $s > \frac{3}{2}$ and $u(t,x)$ be a solution of the Cauchy problem. The solution blows up in a finite time $T \in (0,\infty)$ in the sense of $\lim_{t\uparrow T} \|u(t,\cdot)\|_{H^s} = \infty$ if and only if

$$\liminf_{t\uparrow T}\inf_{x\in\mathbb{S}} u_x(t,x) = -\infty \quad \text{while} \quad \limsup_{t\uparrow T}\sup_{x} |u(t,x)| < \infty.$$

- We have

$$|\partial_x^{-1} u(t,x)| \leq \int_{\mathbb{S}} |u(t,x)|dx \leq \|u\|_{L^2} = \|u\|_{L^2}$$

and

$$\sup_{s\in[0,t]} \|u(s,\cdot)\|_{L^\infty} \leq \|u_0\|_{L^\infty} + t\|u_0\|_{L^2}, \quad \forall t \in [0,T).$$

Theorem

Assume that $u_0(x) \in H^s(\mathbb{S})$, $s > \frac{3}{2}$ *and* $\int_{\mathbb{S}} u_0(x)\,dx = 0$. *If either*

$$\int_{\mathbb{S}} \left(u_0'(x)\right)^3\,dx < -\left(\frac{3}{2}\|u_0\|_{L^2}\right)^{3/2}, \tag{1}$$

or

$$\int_{\mathbb{S}} \left(u_0'(x)\right)^3\,dx < 0 \quad \text{and} \quad \|u_0\|_{L^2} > \frac{3}{4}, \tag{2}$$

or there is a $x_0 \in \mathbb{S}$ *such that*

$$u_0'(x_0) \leq -(1+\epsilon)\left(\|u_0\|_{L^\infty} + T_1\|u_0\|_{L^2}\right)^{\frac{1}{2}}, \tag{3}$$

where T_1 *is the smallest positive root of*

$$2T_1\left(\|u_0\|_{L^\infty} + T_1\|u_0\|_{L^2}\right)^{\frac{1}{2}} = \log\left(1 + \frac{2}{\varepsilon}\right),$$

then the solution $u(t,x)$ *of the Cauchy problem blows up in a finite time.*

Proof of sufficient condition (1)

Direct computation gives

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{S}} u_x^3\,dx &= 3\int_{\mathbb{S}} u_x^2\left(-u_x^2 - uu_{xx} + u\right)\,dx \\ &= -2\int_{\mathbb{S}} u_x^4\,dx + 3\int_{\mathbb{S}} uu_x^2\,dx \\ &\leq -2\|u_x\|_{L^4}^4 + 3\|u\|_{L^2}\|u_x\|_{L^4}^2. \end{aligned}$$

By Hölder's inequality, we have

$$|V(t)| \leq \|u_x\|_{L^3}^3 \leq \|u_x\|_{L^4}^3, \quad V(t) = \int_{\mathbb{S}} u_x^3(t,x)\,dx < 0.$$

Let $Q_0 = \|u\|_{L^2}^2 = \|u_0\|_{L^2}^2$ and $V(0) < -\left(\frac{3}{2}Q_0\right)^{\frac{3}{2}}$. Then,

$$\frac{dV}{dt} \leq -2\left(|V|^{\frac{2}{3}} - \frac{3Q_0}{4}\right)^2 + \frac{9Q_0^2}{8},$$

There is $T < \infty$ such that $V(t) \to -\infty$ as $t \uparrow T$.

Proof of sufficient condition (3)

Let $\xi \in \mathbb{S}$, $t \in [0, T)$, and denote

$$x = X(\xi, t), \quad u(x,t) = U(\xi, t), \quad \partial_x^{-1} u(x,t) = G(\xi, t).$$

At characteristics $x = X(\xi, t)$, we obtain

$$\begin{cases} \dot{X}(t) = U, \\ X(0) = \xi, \end{cases} \qquad \begin{cases} \dot{U}(t) = G, \\ U(0) = u_0(\xi), \end{cases}$$

- The map $X(\cdot, t) : \mathbb{S} \mapsto \mathbb{R}$ is an increasing diffeomorphism with

$$\partial_\xi X(\xi, t) = \exp\left(\int_0^t u_x(X(\xi, s), s) ds \right) > 0, \;\; t \in [0, T), \;\; \xi \in \mathbb{S}.$$

- Using

$$U(t, \xi) = u_0(\xi) + \int_0^t G(s, \xi) ds, \quad t \in [0, T),$$

  we obtain

$$\sup_{s \in [0,t]} \sup_{\xi \in \mathbb{S}} |U(s, \xi)| \leq \|u_0\|_{L^\infty} + t \|u_0\|_{L^2}, \quad t \in [0, T).$$

Proof of sufficient condition (3)

Let $V(\xi, t) = u_x(t, X(\xi, t))$. Then

$$\dot{V} = -V^2 + U \quad \Rightarrow \quad \dot{V} \leq -V^2 + (\|u_0\|_{L^\infty} + \gamma t \|u_0\|_{L^2})$$

If there is a $x_0 \in \mathbb{S}$ such that

$$V(0) \leq -(1+\epsilon)\left(\|u_0\|_{L^\infty} + T_1 \|u_0\|_{L^2}\right)^{\frac{1}{2}},$$

where T_1 is the smallest positive root of

$$2T_1 \left(\|u_0\|_{L^\infty} + T_1 \|u_0\|_{L^2}\right)^{\frac{1}{2}} = \log\left(1 + \frac{2}{\varepsilon}\right),$$

then $V(t) \to -\infty$ as $t \uparrow T < T_1$.

Remarks

- If $\epsilon \to \infty$, then $T \to 0$. The steeper the slope of $u_0(x)$, the quicker the solution $u(t,x)$ blows up.
- If $u_0 \in H^3(\mathbb{S})$ and $T < \infty$ is the blow-up time, then

$$\lim_{t \uparrow T} (T-t) \inf_{x \in \mathbb{S}} u_x(t,x) = -1, \quad \lim_{t \uparrow T} (T-t) \sup_{x \in \mathbb{S}} u_x(t,x) = 0.$$

- Blow-up results can be extended on an infinite line in space $u(t) \in C([0,T); H^s(\mathbb{R}) \cap \dot{H}^{-1}(\mathbb{R}))$, where $\dot{H}^{-1}(\mathbb{R})$ is needed for the energy conservation

$$E = \int_{\mathbb{R}} \left(\frac{1}{3} u^3 + (\partial_x^{-1} u)^2 \right) dx,$$

and control of L^∞-norm

$$\sup_{s \in [0,t]} \|u(s,\cdot)\|_{L^\infty} \leq \|u_0\|_{L^\infty} + Ct + \frac{1}{6} \|u_0\|_{L^2}^2 t^2, \quad t \in [0,T).$$

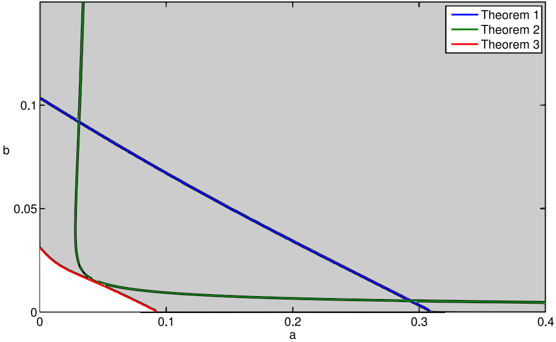
Numerical simulation

Using the pseudospectral method, we solve

$$\frac{\partial}{\partial t}\hat{u}_k = -\frac{i}{k}\hat{u}_k - \frac{ik}{2}\mathcal{F}\left[\left(\mathcal{F}^{-1}\hat{u}\right)^2\right]_k, \quad k \neq 0, \quad t > 0.$$

Consider the 1-periodic initial data

$$u_0(x) = a\cos(2\pi x) + b\sin(4\pi x),$$



Evolution of the cosine initial data

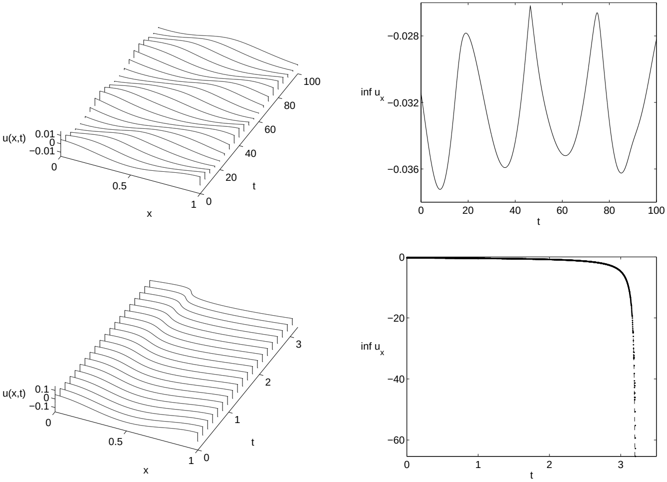


Figure: Solution surface $u(t,x)$ (left) and $\inf_{x\in\mathbb{S}} u_x(t,x)$ versus t (right) for $a = 0.005$, $b = 0$ (top) and $a = 0.05$, $b = 0$ (bottom). $C \approx -1.009$ and $B \approx 3.213$.

Evolution of the cosine-sine initial data

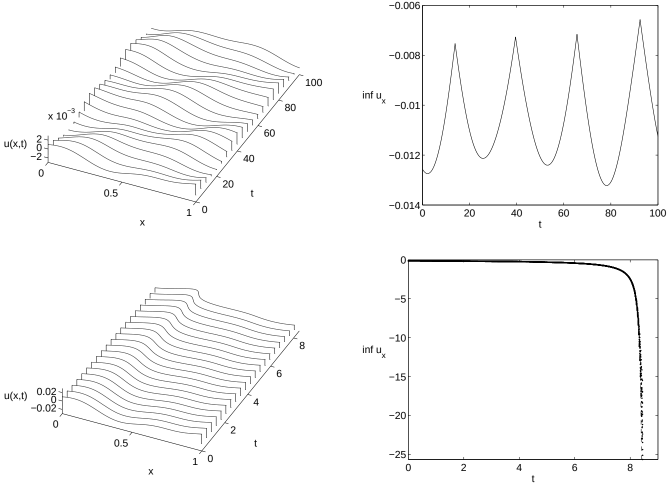


Figure: The same as Figure 1 but for $a = 0.001$, $b = 0.0005$ (top) and $a = 0.01$, $b = 0.005$ (bottom). The least squares fit is computed with $C \approx -1.042$ and $B \approx 8.442$.

Power fit

We compute the best power fit for

$$V(t) := \inf_{x \in \mathbb{S}} u_x(t, x)$$

according to the blow-up law

$$V(t) \simeq \frac{-1}{B + Ct} \quad \text{for} \quad 0 < T - t \ll 1.$$

Note that the analytical blow-up result,

$$\lim_{t \uparrow T} (T - t) \inf_{x \in \mathbb{S}} u_x(t, x) = -1,$$

implies that $C = -1$.