Global existence and wave breaking in the short-pulse and Ostrovsky–Hunter equations

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References:

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Ostrovsky equation for rotating fluid

The **Ostrovsky equation** is a model for small-amplitude long waves in a rotating fluid of a finite depth [Ostrovsky, 1978]:

$$(u_t + uu_x - \beta u_{xxx})_x = \gamma u,$$

where β and γ are real coefficients.

When $\beta = 0$ and $\gamma = 1$, the Ostrovsky equation is

$$(u_t + uu_x)_x = u,$$

and is known under the names of

- the short-wave equation [Hunter, 1990];
- Ostrovsky–Hunter equation [Boyd, 2005];
- reduced Ostrovsky equation [Stepanyants, 2006];
- the Vakhnenko equation [Vakhnenko & Parkes, 2002].

Short-pulse equation

The **short-pulse equation** is a model for propagation of ultra-short pulses with few cycles on the pulse scale [Schäfer, Wayne 2004]:

$$u_{xt} = u + \frac{1}{6}\left(u^3\right)_{xx},$$

where all coefficients are normalized thanks to the scaling invariance.

The short-pulse equation

- replaces the nonlinear Schrödinger equation for short wave packets
- features exact solutions for modulated pulses
- enjoys inverse scattering and an infinite set of conserved quantities

Relevant results

- T. Schafer and C.E. Wayne (2004) proved local existence in $H^2(\mathbb{R})$.
- A. Stefanov *et al.* (2010) considered a family of the generalized short-pulse equations
$$u_{xt} = u + (u^p)_{xx}$$
and proved scattering to zero for *small* initial data if $p \geq 4$.
- We proved both global well-posedness for *small* initial data and wave breaking for *large* initial data if $p = 3$.
- We proved wave breaking for sufficiently *large* initial data if $p = 2$ but found no proof of global existence for *small* initial data.
- C. Holliman (the group of A. Himonas) (2010-2011) proved the lack of uniform continuity with respect to initial data for a number of equations, including the Ostrovsky–Hunter and Hunter-Saxton equations,
$$(u_t + uu_x)_x = (u_x)^2.$$

Integrability of the short-pulse equation

Let $x = x(y,t)$ satisfy

$$\begin{cases} x_y = \cos w, \\ x_t = -\frac{1}{2} w_t^2. \end{cases}$$

Then, $w = w(y,t)$ satisfies the sine–Gordon equation in characteristic coordinates [A. Sakovich, S. Sakovich, J. Phys. A **39**, L361 (2006)]:

$$w_{yt} = \sin(w).$$

Lemma

Let the mapping $[0,T] \ni t \mapsto w(\cdot,t) \in H_c^s$ *be* C^1 *and*

$$H_c^s = \left\{ w \in H^s(\mathbb{R}) : \quad \|w\|_{L^\infty} \le w_c < \frac{\pi}{2} \right\}, \quad s \ge 1.$$

Then, $x(y,t)$ *is invertible in* y *for any* $t \in [0,T]$ *and* $u(x,t) = w_t(y(x,t),t)$ *solves the short-pulse equation*

$$u_{xt} = u + \frac{1}{6}\left(u^3\right)_{xx}, \quad x \in \mathbb{R}, \quad t \in [0,T].$$

Solutions of the short-pulse equation

A kink of the sine–Gordon equation gives a *loop solution* of the short-pulse equation:

$$\begin{cases} u = 2\,\mathrm{sech}(y+t), \\ x = y - 2\tanh(y+t). \end{cases}$$

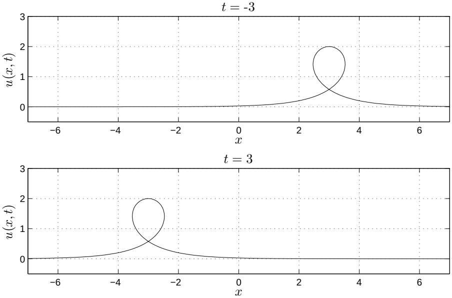


Figure: The loop solution $u(x,t)$ to the short-pulse equation

Solutions of the short-pulse equation

A breather of the sine–Gordon equation gives a *pulse solution* of the short-pulse equation:

$$\left\{ \begin{array}{l} u(y,t) = 4mn\dfrac{m\sin\psi\sinh\phi + n\cos\psi\cosh\phi}{m^2\sin^2\psi + n^2\cosh^2\phi} = u\left(y - \frac{\pi}{m}, t + \frac{\pi}{m}\right), \\ x(y,t) = y + 2mn\dfrac{m\sin 2\psi - n\sinh 2\phi}{m^2\sin^2\psi + n^2\cosh^2\phi} = x\left(y - \frac{\pi}{m}, t + \frac{\pi}{m}\right) + \frac{\pi}{m}, \end{array} \right.$$

where

$$\phi = m(y+t), \quad \psi = n(y-t), \quad n = \sqrt{1-m^2},$$

and $m \in \mathbb{R}$ is a free parameter.

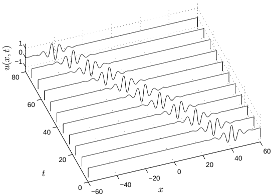


Figure: The pulse solution to the short-pulse equation with $m = 0.25$

Local well-posedness of the short-pulse equation

Theorem (Schäfer & Wayne, 2004)

Let $u_0 \in H^2$. There exists a maximal existence time $T = T(u_0) > 0$ and a unique solution to the short-pulse equation

$$u(t) \in C([0,T), H^2) \cap C^1([0,T), H^1)$$

that satisfies $u(0) = u_0$ and depends continuously on u_0.

Remarks:

- The proof can be extended to any $s > \frac{3}{2}$ (Stefanov *et al*, 2010).
- There is a constraint on solutions of the short-pulse equation

$$\int_{\mathbb{R}} u(x,t)dx = 0, \quad t > 0.$$

Consider the Cauchy problem for the sine-Gordon equation

$$\begin{cases} w_{yt} = \sin w, & y \in \mathbb{R}, \quad t > 0 \\ w|_{t=0} = w_0, & y \in \mathbb{R}. \end{cases}$$

Note: if $w \in C^1([0,T), H^s(\mathbb{R}))$, $s > \frac{1}{2}$, then

$$\int_{\mathbb{R}} \sin w(y,t) dy = 0, \quad t \in (0,T).$$

The standard method of Picard–Kato would not work because if $w(\cdot,t) \in H^s$, $s > \frac{1}{2}$, then $\sin(w(\cdot,t)) \in H^s$, but $\partial_y^{-1} \sin(w(y,t))dy$ may not be in H^s.

Let $q = \sin(w)$ and rewrite the Cauchy problem in the equivalent form

$$\begin{cases} q_t = (1 - f(q))\partial_y^{-1} q, \\ q|_{t=0} = q_0, \end{cases}$$

where

$$f(q) := 1 - \sqrt{1 - q^2} = \frac{q^2}{1 + \sqrt{1 - q^2}}, \quad \forall |q| \leq 1 : \quad \frac{q^2}{2} \leq f(q) \leq q^2.$$

Consider the initial-value problem

$$\begin{cases} q_t = (1 - f(q))\partial_y^{-1} q, \\ q|_{t=0} = q_0. \end{cases}$$

Now the constraints are

$$\|q(\cdot,t)\|_{L^\infty} < 1, \quad \int_{\mathbb{R}} q(y,t)dy = 0, \quad t > 0.$$

Theorem

Assume that $q_0 \in X_c^s$, $s > \frac{1}{2}$, where

$$X_c^s = \left\{ q \in H^s \cap \dot{H}^{-1}, \ \|q\|_{L^\infty} \leq q_c < 1 \right\}.$$

There exist a maximal time $T = T(q_0) > 0$ and a unique solution $q(t) \in C([0,T), X_c^s)$ of the Cauchy problem that satisfies $q(0) = q_0$ and depends continuously on q_0.

Duhamel's method

Consider the Cauchy problem for the linearized sine–Gordon equation

$$\begin{cases} Q_t = \partial_y^{-1} Q, \\ Q|_{t=0} = Q_0. \end{cases}$$

Denote

$$L = \partial_y^{-1} \quad \text{and} \quad Q(t) = e^{tL} Q_0.$$

The solution operator e^{tL} is an *isometry* from H^s to H^s for any $s \geq 0$, so that

$$\|Q(t)\|_{H^s} = \|e^{tL} Q_0\|_{H^s} = \|Q_0\|_{H^s}, \quad \forall t \in \mathbb{R}.$$

By Duhamel's principle, we have

$$q(t) = e^{tL} q_0 - \int_0^t e^{(t-t')L} f(q(t'))\, \partial_y^{-1} q \, dt'.$$

Sketch of the proof

Fix $q_c \in (0,1)$, $\delta > 0$ and $\alpha \in (0,1)$ so that the initial data satisfy

$$\|q_0\|_{X^s} \leq \alpha\delta, \quad \|q_0\|_{L^\infty} \leq \alpha q_c$$

We need to show that there exists $T > 0$ such that

- the mapping

$$(Aq)(t) = \int_0^t e^{(t-t')L} f(q(t'))\, \partial_y^{-1} q\, dt' : \quad C([0,T], X_c^s) \mapsto C([0,T], X_c^s)$$

  is Lipschitz continuous and a contraction for sufficiently small $T > 0$.

- The integral equation is well-defined in

$$\|q(t)\|_{X^s} \leq \delta, \quad \|q(t)\|_{L^\infty} \leq q_c, \quad t \in [0,T].$$

Existence, uniqueness, and continuous dependence come from the standard Banach's Fixed-Point Theorem.

Our local well-posedness of the short-pulse equation

Theorem (P., Sakovich, 2010)

Let $u_0 \in H^s \cap \dot{H}^{-1}$, $s > 3/2$. *There exists a maximal existence time* $T = T(u_0) > 0$ *and a unique solution to the short-pulse equation*

$$u(t) \in C^1([0,T), H^s \cap \dot{H}^{-1})$$

that satisfies $u(0) = u_0$ *and depends continuously on* u_0.

This theorem follows from the local well-posedness of the sine–Gordon equation and the correspondence

$$u = w_t = \frac{q_t}{\sqrt{1-q^2}} = p, \quad u_x = \frac{w_{ty}}{\cos(w)} = \tan(w) = \frac{p_y}{\sqrt{1-q^2}}.$$

Conserved quantities of the short-pulse equation

A bi-infinite hierarchy of conserved quantities of the short-pulse equation was found in Brunelli [J.Math.Phys. **46**, 123507 (2005)]:

$$
\begin{aligned}
&\dots \\
E_{-1} &= \int_{\mathbb{R}} \left(\frac{1}{24} u^4 - \frac{1}{2} (\partial_x^{-1} u)^2 \right) dx, \\
E_0 &= \int_{\mathbb{R}} u^2 dx, \\
E_1 &= \int_{\mathbb{R}} \frac{u_x^2}{1 + \sqrt{1 + u_x^2}} dx, \\
E_2 &= \int_{\mathbb{R}} \frac{u_{xx}^2}{(1 + u_x^2)^{5/2}} dx, \\
&\dots
\end{aligned}
$$

Global well-posedness of the short-pulse equation

Theorem (P. & Sakovich, 2010)

Let $u_0 \in H^2$ and the conserved quantities satisfy $2E_1 + E_2 < 1$. Then the short-pulse equation admits a unique solution $u(t) \in C(\mathbb{R}_+, H^2)$ with $u(0) = u_0$.

The values of E_0, E_1 and E_2 are bounded by $\|u_0\|_{H^2}$ as follows:

$$\begin{aligned} E_0 &= \int_{\mathbb{R}} u^2 dx = \|u_0\|_{L^2}^2, \\ E_1 &= \int_{\mathbb{R}} \frac{u_x^2}{1+\sqrt{1+u_x^2}} dx \leq \frac{1}{2}\|u_0'\|_{L^2}^2, \\ E_2 &= \int_{\mathbb{R}} \frac{u_{xx}^2}{(1+u_x^2)^{5/2}} dx \leq \|u_0''\|_{L^2}^2. \end{aligned}$$

The existence time $T > 0$ of the local solutions is inverse proportional to the norm $\|u_0\|_{H^2}$ of the initial data. To extend T to ∞, we need to control the norm $\|u(t)\|_{H^2}$ by a T-independent constant on $[0, T]$.

Sketch of the proof

- Let $\tilde{q}(x,t) = \frac{u_x}{\sqrt{1+u_x^2}}$. Then, we obtain

$$\|\tilde{q}(t)\|_{H^1} \leq \sqrt{2E_1 + E_2} < 1, \quad t \in [0,T).$$

- Thanks to Sobolev's embedding $\|\tilde{q}\|_{L^\infty} \leq \frac{1}{\sqrt{2}}\|\tilde{q}\|_{H^1} < 1$, so that $u_x = \frac{\tilde{q}}{\sqrt{1-\tilde{q}^2}}$ satisfies the bound

$$\|u_x(t)\|_{H^1} \leq \frac{\|\tilde{q}\|_{H^1}}{\sqrt{1 - \|\tilde{q}\|_{H^1}^2}}, \quad t \in [0,T)$$

or equivalently

$$\|u(t)\|_{H^2} \leq \left(E_0 + \frac{2E_1 + E_2}{1 - (2E_1 + E_2)}\right)^{1/2}, \quad t \in [0,T).$$

Corollary

Let $u_0 \in H^2$ such that $2\sqrt{2E_1E_2} < 1$. Then the short-pulse equation admits a unique solution $u(t) \in C(\mathbb{R}_+, H^2)$ with $u(0) = u_0$.

Let $\alpha \in \mathbb{R}_+$ be an arbitrary parameter. If $u(x,t)$ is a solution of the short-pulse equation, then $U(X,T)$ is also a solution with

$$X = \alpha x, \quad T = \alpha^{-1} t, \quad U(X,T) = \alpha u(x,t).$$

The scaling invariance yields transformation $\tilde{E}_1 = \alpha E_1$ and $\tilde{E}_2 = \alpha^{-1} E_2$. For a given $u_0 \in H^2$, a family of initial data $U_0 \in H^2$ satisfies

$$\phi(\alpha) = 2\tilde{E}_1 + \tilde{E}_2 = 2\alpha E_1 + \alpha^{-1} E_2 \geq 2\sqrt{2E_1E_2}, \quad \forall \alpha \in \mathbb{R}_+.$$

If $2\sqrt{2E_1E_2} < 1$, there exists α such that $U(X,T)$ is defined for any $T \in \mathbb{R}_+$.

Short-pulse equation in a periodic domain

Let $\mathbb{S}$ be the unit circle and let ∂_x^{-1} be the mean-zero anti-derivative

$$\partial_x^{-1} u = \int_0^x u(x', t) dx' - \int_{\mathbb{S}} \int_0^x u(x', t) dx' dx.$$

The short-pulse equation on a circle is given by

$$\begin{cases} u_t = \frac{1}{2} u^2 u_x + \partial_x^{-1} u, \\ u(x, 0) = u_0(x), \end{cases} \qquad x \in \mathbb{S}, \quad t \geq 0.$$

Let $u(t) \in C([0, T), H^s(\mathbb{S})) \cap C^1([0, T), H^{s-1}(\mathbb{S}))$ be a local solution such that $u(0) = u_0 \in H^s(\mathbb{S})$.

- The assumption $\int_{\mathbb{S}} u_0(x) dx = 0$ is necessary for existence.
- The following quantities are constant on $[0, T)$:

$$E_0 = \int_{\mathbb{S}} u^2 dx, \quad E_1 = \int_{\mathbb{S}} \sqrt{1 + u_x^2} dx$$

Finite-time blow-up scenario

Lemma

Let $u_0 \in H^2(\mathbb{S})$ *and* $u(t)$ *be a local solution of the Cauchy problem. The solution blows up in a finite time* $T < \infty$ *in the sense* $\lim_{t\uparrow T} \|u(\cdot,t)\|_{H^2} = \infty$ *if and only if*

$$\lim_{t\uparrow T} \sup_{x\in\mathbb{S}} u(x,t)u_x(x,t) = +\infty.$$

For the inviscid Burgers equation

$$\begin{cases} u_t = \frac{1}{2}u^2 u_x, \\ u(x,0) = u_0(x), \end{cases} \qquad x \in \mathbb{S}, \quad t \geq 0.$$

the problem can be solved by the method of characteristics. The finite-time blow-up occurs for any $u_0(x) \in C^1(\mathbb{S})$ if there is a point $x_0 \in \mathbb{S}$ such that $u_0(x_0)u_0'(x_0) > 0$. The blow-up time is

$$T = \inf_{\xi\in\mathbb{S}} \left\{ \frac{1}{u_0(\xi)u_0'(\xi)} : \quad u_0(\xi)u_0'(\xi) > 0 \right\}.$$

Method of characteristics

Let $\xi \in \mathbb{S}$, $t \in [0, T)$, and denote

$$x = X(\xi, t), \quad u(x,t) = U(\xi, t), \quad \partial_x^{-1} u(x,t) = G(\xi, t).$$

At characteristics $x = X(\xi, t)$, we obtain

$$\begin{cases} \dot{X}(t) = -\frac{1}{2} U^2, \\ X(0) = \xi, \end{cases} \qquad \begin{cases} \dot{U}(t) = G, \\ U(0) = u_0(\xi), \end{cases}$$

- The map $X(\cdot, t) : \mathbb{S} \mapsto \mathbb{R}$ is an increasing diffeomorphism with

$$\partial_\xi X(\xi, t) = \exp\left(\int_0^t u(X(\xi,s),s) u_x(X(\xi,s),s) ds \right) > 0, \;\; t \in [0,T), \;\; \xi \in \mathbb{S}.$$

- The following quantities are bounded on $[0, T)$:

$$|u(x,t)| \leq \left| \int_{\xi_t}^x u_x(x,t)\, dx \right| \leq \int_{\mathbb{S}} |u_x(x,t)| dx \leq E_1$$

and

$$|\partial_x^{-1} u(x,t)| \leq \left| \int_{\tilde{\xi}_t}^x u(x,t)\, dx \right| \leq \int_{\mathbb{S}} |u(x,t)| dx \leq \sqrt{E_0}.$$

Sufficient condition for wave breaking

Theorem (Liu, P. & Sakovich, 2009)

Let $u_0 \in H^2(\mathbb{S})$ *and* $\int_{\mathbb{S}} u_0(x)\,dx = 0$. *Assume that there exists* $x_0 \in \mathbb{R}$ *such that* $u_0(x_0)u_0'(x_0) > 0$ *and*

$$\text{either} \qquad |u_0'(x_0)| > \left(\frac{E_1^2}{4E_0^{1/2}}\right)^{1/3},$$

$$|u_0(x_0)||u_0'(x_0)|^2 > E_1 + \left(2E_0^{1/2}|u_0'(x_0)|^3 - \frac{1}{2}E_1^2\right)^{1/2},$$

$$\text{or} \qquad |u_0'(x_0)| \le \left(\frac{E_1^2}{4E_0^{1/2}}\right)^{1/3}, \quad |u_0(x_0)||u_0'(x_0)|^2 > E_1.$$

Then there exists a finite time $T \in (0,\infty)$ *such that the solution* $u(t) \in C([0,T), H^2(\mathbb{S}))$ *of the Cauchy problem blows up with the property*

$$\lim_{t\uparrow T} \sup_{x\in\mathbb{S}} u(x,t)u_x(x,t) = +\infty, \quad \textit{while} \quad \lim_{t\uparrow T} \|u(\cdot,t)\|_{L^\infty} \le E_1.$$

Sketch of the proof

Let $V(\xi,t) = u_x(X(\xi,t),t)$ and $W(\xi,t) = U(\xi,t)V(\xi,t)$. Then

$$\begin{cases} \dot{V} &= VW + U, \\ \dot{W} &= W^2 + VG + U^2. \end{cases}$$

Under the conditions of the theorem, there exists $\xi_0 \in \mathbb{S}$ such that $V(\xi_0,t)$ and $W(\xi_0,t)$ satisfy the apriori estimates

$$\begin{cases} \dot{V} &\geq VW - E_1, \\ \dot{W} &\geq W^2 - V\sqrt{E_0}. \end{cases}$$

We show that $V(\xi_0,t)$ and $W(\xi_0,t)$ go to infinity in a finite time.

Criteria of well-posedness and wave breaking

Consider Gaussian initial data

$$u_0(x) = a(1 - 2bx^2)e^{-bx^2}, \quad x \in \mathbb{R},$$

where (a, b) are arbitrary and $\int_{\mathbb{R}} u_0(x)dx = 0$ is satisfied.

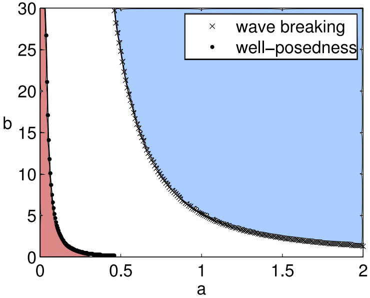


Figure: Global solutions exist in the red region and wave breaking occurs in the blue region.

Numerical simulation

Using the pseudospectral method, we solve

$$\frac{\partial}{\partial t}\hat{u}_k = -\frac{i}{k}\hat{u}_k + \frac{ik}{6}\mathcal{F}\left[\left(\mathcal{F}^{-1}\hat{u}\right)^3\right]_k, \quad k \neq 0, \quad t > 0.$$

Consider the 1-periodic initial data

$$u_0(x) = a\cos(2\pi x)$$

- Criterion for wave breaking: $a > 1.053$.
- Criterion for global solutions: $a < 0.0354$.

Evolution of the cosine initial data

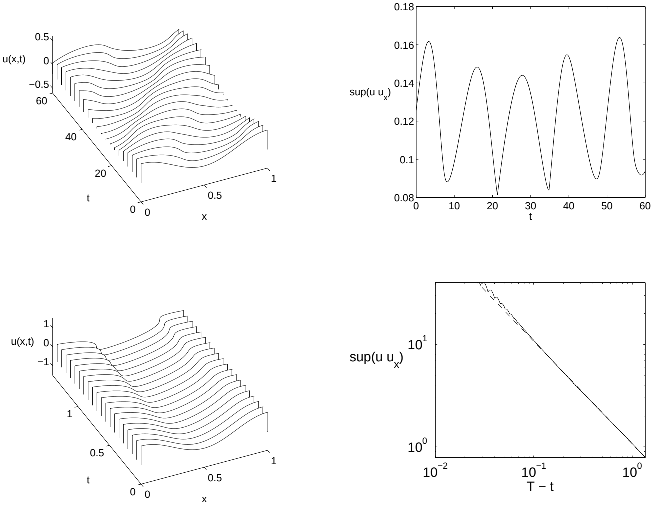


Figure: Solution surface $u(x,t)$ (left) and the supremum norm $W(t)$ (right) for $a = 0.2$ (top) and $a = 0.5$ (bottom).

Power fit

We compute the best power fit for

$$W(t) := \sup_{x \in \mathbb{S}} u(x,t)u_x(x,t)$$

according to the blow-up law

$$W(t) \simeq \frac{C}{T-t} \quad \text{for} \quad 0 < T - t \ll 1.$$

Note that the inviscid Burgers equation has the exact blow-up law $W(t) = \frac{1}{T-t}$.

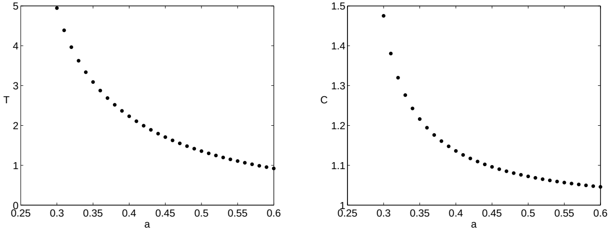


Figure: Time of wave breaking T versus a (left). Constant C of the linear regression versus a (right).

Cauchy problem on a circle $\mathbb{S}$ of unit length:

$$\begin{cases} u_t + uu_x = \partial_x^{-1} u, & t > 0, \\ u(0,x) = u_0(x), \end{cases}$$

where

$$\partial_x^{-1} u := \int_0^x u(t,x')dx' - \int_{\mathbb{S}} \int_0^x u(t,x')dx'dx.$$

The inviscid Burgers equation $u_t + uu_x = 0$ develops wave breaking in a finite time for any initial data $u(0,x) = u_0(x)$ if $u_0(x) \in C^1$ and there is a point x_0 such that $u_0'(x_0) < 0$. In other words, there exists a finite time $T \in (0,\infty)$ such that

$$\liminf_{t \uparrow T} \inf_x u_x(t,x) = -\infty, \quad \text{while} \quad \limsup_{t \uparrow T} \sup_x |u(t,x)| < \infty.$$

Moreover, the blow-up time is computed by the method of characteristics:

$$T = \inf_{\xi} \left\{ \frac{1}{|u_0'(\xi)|} : \quad u_0'(\xi) < 0 \right\}.$$

Conserved quantities

For the Ostrovsky–Hunter equation, it was found that

Theorem (Hunter, 1990)

Let $u_0(x) \in C^1(\mathbb{S})$*, where* $\mathbb{S}$ *is a circle of unit length, and define*

$$\inf_{x\in\mathbb{S}} u_0'(x) = -m \quad \text{and} \quad \sup_{x\in\mathbb{S}} |u_0(x)| = M.$$

If $m^3 > 4M(4+m)$*, a smooth solution* $u(t,x)$ *breaks down at a finite time.*

Note that there exist infinitely many conserved quantities of the Ostrovsky-Hunter equations, which are not useful:

$$\begin{aligned} E_0 &= \int_{\mathbb{R}} u^2 dx, \\ E_{-1} &= \int_{\mathbb{R}} \left(\frac{1}{3} u^3 + (\partial_x^{-1} u)^2 \right) dx, \\ \ldots \end{aligned}$$

Finite-time wave breaking

- Let $u_0(x) \in H^s(\mathbb{S})$, $s > \frac{3}{2}$ and $u(t,x)$ be a solution of the Cauchy problem. The solution blows up in a finite time $T \in (0, \infty)$ in the sense of $\lim_{t \uparrow T} \|u(t,\cdot)\|_{H^s} = \infty$ if and only if

$$\liminf_{t \uparrow T} \inf_{x \in \mathbb{S}} u_x(t,x) = -\infty \quad \text{while} \quad \limsup_{t \uparrow T} \sup_x |u(t,x)| < \infty.$$

- We have

$$|\partial_x^{-1} u(t,x)| \leq \int_{\mathbb{S}} |u(t,x)| dx \leq \|u\|_{L^2} = \|u\|_{L^2}$$

and

$$\sup_{s \in [0,t]} \|u(s,\cdot)\|_{L^\infty} \leq \|u_0\|_{L^\infty} + t\|u_0\|_{L^2}, \quad \forall t \in [0,T).$$

Theorem

Assume that $u_0(x) \in H^s(\mathbb{S})$, $s > \frac{3}{2}$ *and* $\int_{\mathbb{S}} u_0(x)\,dx = 0$. *If either*

$$\int_{\mathbb{S}} \left(u_0'(x)\right)^3 \, dx < -\left(\frac{3}{2}\|u_0\|_{L^2}\right)^{3/2}, \tag{1}$$

or

$$\int_{\mathbb{S}} \left(u_0'(x)\right)^3 \, dx < 0 \quad \text{and} \quad \|u_0\|_{L^2} > \frac{3}{4}, \tag{2}$$

or there is a $x_0 \in \mathbb{S}$ *such that*

$$u_0'(x_0) \leq -(1+\epsilon)\left(\|u_0\|_{L^\infty} + T_1\|u_0\|_{L^2}\right)^{\frac{1}{2}}, \tag{3}$$

where T_1 *is the smallest positive root of*

$$2T_1\left(\|u_0\|_{L^\infty} + T_1\|u_0\|_{L^2}\right)^{\frac{1}{2}} = \log\left(1 + \frac{2}{\varepsilon}\right),$$

then the solution $u(t,x)$ *of the Cauchy problem blows up in a finite time.*

Proof of sufficient conditions (1)–(2)

Direct computation gives

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{S}} u_x^3\,dx &= 3\int_{\mathbb{S}} u_x^2\left(-u_x^2 - uu_{xx} + u\right)\,dx \\ &= -2\int_{\mathbb{S}} u_x^4\,dx + 3\int_{\mathbb{S}} uu_x^2\,dx \\ &\leq -2\|u_x\|_{L^4}^4 + 3\|u\|_{L^2}\|u_x\|_{L^4}^2. \end{aligned}$$

By Hölder's inequality, we have

$$|V(t)| \leq \|u_x\|_{L^3}^3 \leq \|u_x\|_{L^4}^3, \quad V(t) = \int_{\mathbb{S}} u_x^3(t,x)\,dx < 0.$$

Let $Q_0 = \|u\|_{L^2}^2 = \|u_0\|_{L^2}^2$ and $V(0) < -\left(\frac{3}{2}Q_0\right)^{\frac{3}{2}}$. Then,

$$\frac{dV}{dt} \leq -2\left(|V|^{\frac{2}{3}} - \frac{3Q_0}{4}\right)^2 + \frac{9Q_0^2}{8},$$

There is $T < \infty$ such that $V(t) \to -\infty$ as $t \uparrow T$.

Proof of sufficient condition (3)

Let $\xi \in \mathbb{S}$, $t \in [0, T)$, and denote

$$x = X(\xi, t), \quad u(x, t) = U(\xi, t), \quad \partial_x^{-1} u(x, t) = G(\xi, t).$$

At characteristics $x = X(\xi, t)$, we obtain

$$\begin{cases} \dot{X}(t) = U, \\ X(0) = \xi, \end{cases} \qquad \begin{cases} \dot{U}(t) = G, \\ U(0) = u_0(\xi), \end{cases}$$

Let $V(\xi, t) = u_x(t, X(\xi, t))$. Then

$$\dot{V} = -V^2 + U \quad \Rightarrow \quad \dot{V} \leq -V^2 + (\|u_0\|_{L^\infty} + \gamma t \|u_0\|_{L^2})$$

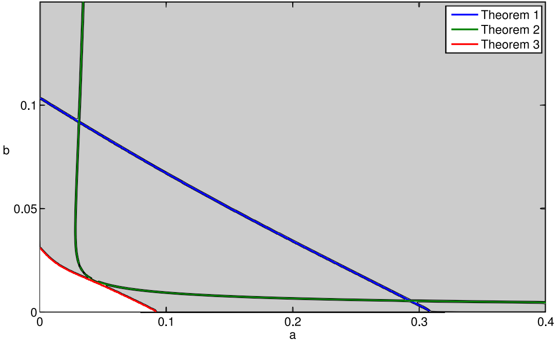
Numerical simulation

Using the pseudospectral method, we solve

$$\frac{\partial}{\partial t}\hat{u}_k = -\frac{i}{k}\hat{u}_k - \frac{ik}{2}\mathcal{F}\left[\left(\mathcal{F}^{-1}\hat{u}\right)^2\right]_k, \quad k \neq 0, \quad t > 0.$$

Consider the 1-periodic initial data

$$u_0(x) = a\cos(2\pi x) + b\sin(4\pi x),$$



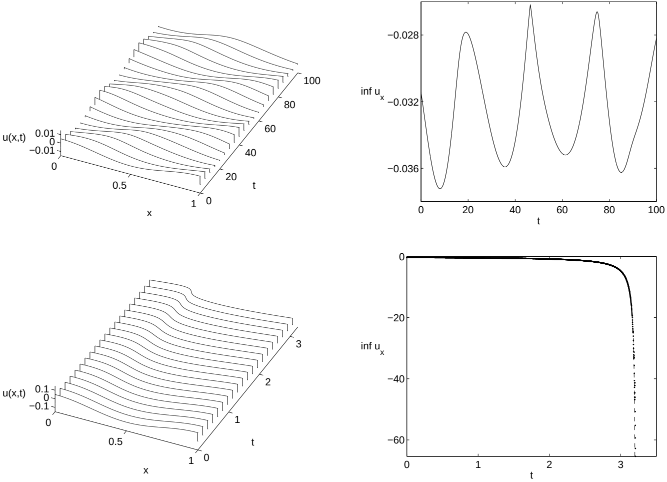


Figure: Solution surface $u(t,x)$ (left) and $\inf_{x\in\mathbb{S}} u_x(t,x)$ versus t (right) for $a = 0.005$, $b = 0$ (top) and $a = 0.05$, $b = 0$ (bottom). $C \approx -1.009$ and $B \approx 3.213$.

Summary

- We found sufficient conditions for global well-posedness of the short-pulse equation for small initial data.
- We found sufficient conditions for wave breaking of the short-pulse and Ostrovsky–Hunter equations for large initial data.
- We illustrated both global existence and wave breaking numerically.
- Numerical results suggest orbital stability of the exact modulated pulses of the short-pulse equation.
- Numerical results suggest global existence for small initial data in the Ostrovsky-Hunter equation.