Nonlinear dynamics in PT-symmetric lattices

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PT-symmetric dNLS equation

We consider the PT-symmetric discrete nonlinear Schrödinger (dNLS) equation

$$i\frac{du_n}{dt} = u_{n+1} + u_{n-1} + i(-1)^n\gamma u_n + |u_n|^2 u_n, \quad n \in S \subset \mathbb{Z},$$

where $\gamma > 0$ is the gain and loss coefficient and $\{u_n\}$ stand for the set of complex amplitudes in the optical network on S.

$S = \{0, 1\}$ gives the optical dimer, $S = \{0, 1, 2, 3\}$ gives the quadrimer, and so on. Generally, the PT-dNLS equation is not a Hamiltonian model with conserved energy for $\gamma \neq 0$.

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Motivations:

- Understand if the dynamics of the PT-symmetric network is globally bounded or unbounded.
- Understand if the stationary solutions exist and remain stable.
- Consider more general network configurations and nonlinearities.

Global existence in $l^2(S)$

For any $n \in S$, the squared amplitude satisfies the evolution equation

$$\frac{d|u_n|^2}{dt} = 2\gamma(-1)^n |u_n|^2 + g_n - g_{n-1},$$

where $g_n := i(u_n \bar{u}_{n+1} - \bar{u}_n u_{n+1})$.

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Adding up all equations on S, we obtain the balance equation

$$\frac{d}{dt} \sum_{n \in S} |u_n|^2 = 2\gamma \sum_{n \in S} (-1)^n |u_n|^2.$$

By Gronwall's inequality, the balance equation results in the a priori bound

$$\sum_{n \in S} |u_n(t)|^2 \leq \left(\sum_{n \in S} |u_n(0)|^2 \right) e^{2\gamma t}, \quad t \in \mathbb{R}.$$

Hence the solution does not blow up in a finite time in $l^2(S)$.

Linear stability of zero equilibrium

Consider the finite and compensated network with $S_N := \{1, 2, ..., 2N\}$. Linearizing at the zero equilibrium and separating $u_n = w_n e^{-iEt}$, we obtain the linear stationary PT-dNLS equation

$$Ew_n = w_{n+1} + w_{n-1} + i\gamma(-1)^n w_n, \quad n \in S_N$$

subject to the Dirichlet end-point conditions $w_0 = w_{2N+1} = 0$.

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Lemma

Eigenvalues of the linear spectral problem are given explicitly:

$$\gamma^2 + E^2 = 4\cos^2\left(\frac{\pi j}{1+2N}\right), \quad 1 \leq j \leq N.$$

In particular, all eigenvalues are simple and real for $\gamma \in (-\gamma_N, \gamma_N)$, *where*

$$\gamma_N := 2\cos\left(\frac{\pi N}{1+2N}\right).$$

Main result

Theorem

- *For every $\gamma \in (0, \gamma_N)$, all solutions of the PT-dNLS equation starting from sufficiently small initial data in $l^2(S_N)$ remain bounded for all times $t \in \mathbb{R}$.*
- *For every $\gamma > 0$, there exist solutions of the PT-dNLS equation starting from sufficiently large initial data in $l^2(S_N)$ which grow exponentially fast as $t \to \infty$.*

These results for the dimer configuration were independently obtained by Picton–Susanto (2013); Barashenkov–Jackson–Flach (2013); and Kevrekidis–P–Tyugin (2013). However, Stokes constants and integrals of motion available for the integrable dimer cannot be generalized for $N > 1$.

Nonlinear dynamics of a dimer: $N = 1$

Consider a PT-symmetric dimer with two complex amplitudes:

$$\left\{ \begin{array}{l} i\frac{da}{dt} = b - i\gamma a + |a|^2 a, \\ i\frac{db}{dt} = a + i\gamma b + |b|^2 b. \end{array} \right.$$

The zero equilibrium is stable if $\gamma \in (-1, 1)$ $(\gamma_1 = 1)$.

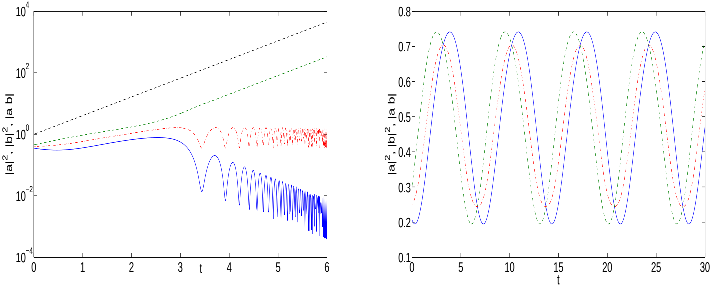

Figure: Left: unbounded growth of the amplitude $|b(t)|$ for the gained oscillator. Right: bounded oscillations of the amplitudes.

Important bound: $N = 1$

Lemma

The product $|a(t)b(t)|$ remains bounded for all times $t \in \mathbb{R}$.

Setting

$$u := \bar{a}b + a\bar{b}, \quad v := i(a\bar{b} - \bar{a}b),$$

we obtain

$$\left\{ \begin{array}{l} \frac{du}{dt} = (|b|^2 - |a|^2)v, \\ \frac{dv}{dt} = (|b|^2 - |a|^2)(2 - u). \end{array} \right.$$

Solving the linear oscillator equation, we obtain

$$\left\{ \begin{array}{l} u(t) = 2 + C_1 \cos\left[\int_0^t (|b|^2 - |a|^2)dt'\right] + C_2 \sin\left[\int_0^t (|b|^2 - |a|^2)dt'\right], \\ v(t) = -C_1 \sin\left[\int_0^t (|b|^2 - |a|^2)dt'\right] + C_2 \cos\left[\int_0^t (|b|^2 - |a|^2)dt'\right], \end{array} \right.$$

where C_1 and C_2 are arbitrary constants. Hence,

$$|a(t)b(t)| \leq 1 + |C_1| + |C_2| \quad \text{for all} \quad t \in \mathbb{R}.$$

Proof of the theorem: $N = 1$

The squared amplitudes $|a|^2$ and $|b|^2$ satisfy the evolution equations:

$$\left\{ \begin{array}{l} \frac{d|a|^2}{dt} = -2\gamma|a|^2 + i(\bar{b}a - b\bar{a}), \\ \frac{d|b|^2}{dt} = 2\gamma|b|^2 - i(\bar{b}a - b\bar{a}). \end{array} \right.$$

Let us prove that $|b(t)|$ may grow to infinity as $t \to \infty$.

Choose the initial data (a_0, b_0) to be sufficiently large so that

$$2\gamma|b_0|^2 - i(\bar{b}_0 a_0 - b_0 \bar{a}_0) \geq 2\gamma|b_0|^2 - 2(1 + |C_1| + |C_2|) > 0.$$

Then, $|b(t)|^2$ will grow and the inequality

$$2\gamma|b(t)|^2 - i(\bar{b}(t)a(t) - b(t)\bar{a}(t)) \geq 2\gamma|b(t)|^2 - 2(1 + |C_1| + |C_2|) > 0,$$

will be preserved for all positive times.

Proof of the theorem: $N = 1$

Hence, we have

$$\frac{d|b|^2}{dt} = 2\gamma|b|^2 - i(\bar{b}a - b\bar{a})$$

and

$$2\gamma|b|^2 - i(\bar{b}a - b\bar{a}) \geq 2\gamma|b|^2 - 2(1 + |C_1| + |C_2|) > 0.$$

From the differential equation,

$$\frac{d|\underline{b}|^2}{dt} = 2\gamma|\underline{b}|^2 - 2(1 + |C_1| + |C_2|) > 0.$$

$|\underline{b}(t)|^2$ grows exponentially like $e^{2\gamma t}$ as $t \to \infty$. By the comparison principle, $|\underline{b}(t)|^2$ is the lower solution for $|b|^2$, hence

$$|b(t)|^2 \geq |\underline{b}(t)|^2 \quad \text{for all} \quad t \geq 0.$$

Hence, $|b(t)|^2$ grows at least exponentially as $t \to \infty$.
At the same time, $|a(t)|^2$ decays as $t \to \infty$.

Proof of the theorem: $N = 1$

Let us write the differential equation in the integral form,

$$|b(t)|^2 = e^{2\gamma t} \left(|b_0|^2 - i \int_0^t e^{-2\gamma\tau} \left[a(\tau)\bar{b}(\tau) - \bar{a}(\tau) b(\tau) \right] d\tau \right).$$

Alternative:

- If
$$|b_0|^2 = i \int_0^\infty e^{-2\gamma t} \left[a(t)\bar{b}(t) - \bar{a}(t) b(t) \right] dt,$$
then $|b(t)|^2$ remains bounded with
$$|b(t)|^2 \leq \gamma^{-1} \sup_{t \in \mathbb{R}_+} |a(t) b(t)|, \quad t \in \mathbb{R}_+$$
- Otherwise, $|b(t)|^2$ grows exactly like $e^{2\gamma t}$.

Proof of the theorem: $N = 1$

Lemma

Let $\gamma \in (0,1)$ be fixed and $\delta := \sqrt{|a_0|^2 + |b_0|^2}$ be small. Then, for every $t_0 = \mathcal{O}(\delta^{-2})$, we have

$$\sup_{t\in[0,t_0]} \sqrt{|a(t)|^2 + |b(t)|^2} \leq \delta e^{2\gamma\delta^2 t}, \quad t \in [0, t_0].$$

The proof uses the following elements:

- Diagonalization of the linear part of the system in normal coordinates.
- Removal of the linear diagonal terms by the phase rotation factors.
- Gronwall's inequality for the purely cubic system.

By a contradiction, if the solution grows, then $|b(t)|^2$ grows like $e^{2\gamma t}$ and this growth contradicts the lemma on the time scale $t = \mathcal{O}(\delta^{-2})$.

Nonlinear dynamics of a quadrimer: $N = 2$

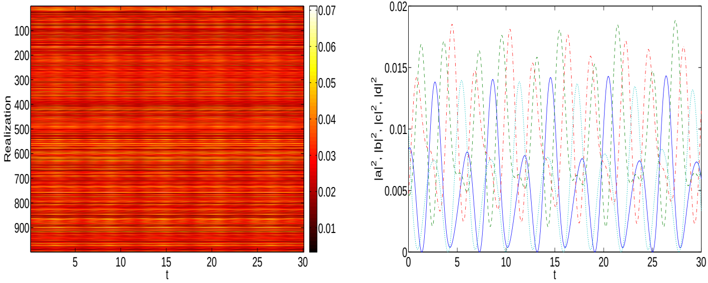

Figure: Dynamics of the quadrimer with small initial data. For the contour plot evolution of the squared l^2 norm, it is clear that all orbits remain bounded (left panel). A typical example of the resulting bounded orbit is shown in the right panel with the blue solid and red dash-dotted lines denoting the gain sites, while the green dashed and cyan dotted lines correspond to the lossy ones.

Nonlinear dynamics of a quadrimer: $N = 2$

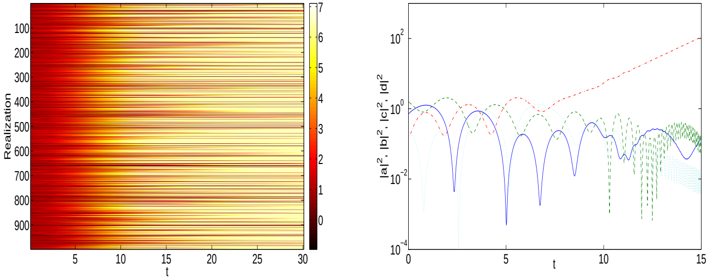

Figure: Dynamics of the quadrimer with large initial data. From the saturation of the left panel's logarithmic scale, it is clear that most trajectories lead to indefinite growth. The right panel illustrates the unbounded dynamics, when one of the gain sites grows.

Generalized PT-symmetric dNLS network

Let us generalize the nonlinear terms in the PT-symmetric dNLS network as

$$\left\{ \begin{array}{l} i\dot{u}_n + v_n + i\gamma u_n + u_{n-1} + u_{n+1} = (|u_n|^2 + |v_n|^2) u_n, \\ i\dot{v}_n + u_n - i\gamma v_n + v_{n-1} + v_{n+1} = (|u_n|^2 + |v_n|^2) v_n, \end{array} \right. \quad n \in S.$$

The zero equilibrium is uniformly stable for any $\gamma \in (-1, 1)$ as the dispersion relation of the linear perturbations is

$$\omega_{\pm}(k) = 2\cos(k) \pm \sqrt{1-\gamma^2}, \quad k \in T_S.$$

The generalized dimer corresponds to $S = \{0\}$.

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- *For every $\gamma > 1$, there exist solutions of the generalized dimer which grow exponentially fast as $t \to \infty$.*

Nonlinear dynamics of the generalized dimer

The generalized dimer can be written explicitly as

$$\left\{ \begin{array}{l} i\frac{da}{dt} = b - i\gamma a + (|a|^2 + |b|^2)a, \\ i\frac{db}{dt} = a + i\gamma b + (|a|^2 + |b|^2)b. \end{array} \right.$$

In Stokes variables,

$$S := |a|^2 + |b|^2, \quad X := |a|^2 - |b|^2, \quad Y := i(\bar{a}b - a\bar{b}), \quad Z := \bar{a}b + a\bar{b},$$

we obtain the system

$$\dot{S} = 2\gamma X, \quad \dot{X} = 2\gamma S - 2Y, \quad \dot{Y} = 2X, \quad \dot{Z} = 0.$$

Hence, Z and $Q := S - \gamma Y$ are constants of motion, whereas S satisfies

$$\ddot{S} + 4(1-\gamma^2)S = 4Q.$$

Another PT-symmetric dNLS network

Another generalized PT-symmetric dNLS network can be written as

$$\left\{ \begin{array}{l} i\dot{u}_n + v_n + i\gamma u_n + u_{n-1} + u_{n+1} = (|u_n|^2 + 2|v_n|^2)u_n + u_n^2 \bar{v}_n, \\ i\dot{v}_n + u_n - i\gamma v_n + v_{n-1} + v_{n+1} = (2|u_n|^2 + |v_n|^2)v_n + v_n^2 \bar{u}_n, \end{array} \right. \quad n \in S.$$

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The zero equilibrium is still uniformly stable for any $\gamma \in (-1,1)$. The generalized dimer corresponds to $S = \{0\}$.

Theorem

For every $\gamma \in \mathbb{R}$, all solutions of the generalized dimer remain bounded for all times $t \in \mathbb{R}$.

Similar results appear back to 1994-1998 in the works of Jorgensen–Christiansen, where a Hamiltonian dimer was studied.

Nonlinear dynamics of a generalized dimer

The generalized dimer is now written explicitly as

$$\begin{cases} i\frac{da}{dt} = b - i\gamma a + (|a|^2 + 2|b|^2)a + a^2\bar{b}, \\ i\frac{db}{dt} = a + i\gamma b + (2|a|^2 + |b|^2)b + b^2\bar{a}. \end{cases}$$

In Stokes variables,

$$S := |a|^2 + |b|^2, \quad X := |a|^2 - |b|^2, \quad Y := i(\bar{a}b - a\bar{b}), \quad Z := \bar{a}b + a\bar{b},$$

we obtain the system

$$\dot{S} = 2\gamma X, \quad \dot{X} = 2\gamma S - 2Y + 2SY, \quad \dot{Y} = 2X - 2SX, \quad \dot{Z} = 0.$$

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Excluding S, we have the planar system

$$\gamma \frac{dY}{dS} = 1 - S,$$

which implies that $Y \sim S^2$ if $S \to \infty$. However, this contradicts to constraint $X^2 + Y^2 + Z^2 = S^2$, hence $|Y| \leq S$.

Nonlinear stationary states

The stationary PT-dNLS equation on the finite network S_N for the PT-symmetric stationary states $u_n(t) = U_n e^{-iEt}$ can be reduced to the system of N algebraic equations

$$EU_n = U_{n+1} + U_{n-1} + i\gamma(-1)^n U_n + |U_n|^2 U_n, \quad 1 \leq n \leq N,$$

subject to the boundary conditions $U_0 = 0$ and $U_{N+1} = \bar{U}_N$.

Theorem

For any $\gamma \in (-1, 1)$, the stationary PT-dNLS equation on S_N for large real E admits 2^N PT-symmetric solutions such that

$$\left| |U_n|^2 - E \right| \leq C, \quad \text{for each } 1 \leq n \leq N.$$

Also it admits exactly 2 solutions such that

$$\left| |U_N|^2 - E \right| \leq C \quad \text{and} \quad |U_n|^2 \leq CE^{-1} \quad \text{for each } 1 \leq n \leq N-1.$$

Stationary states of a dimer: $N = 1$

Setting $E = \frac{1}{\delta}$ and $\mathbf{U} = \frac{\mathbf{W}}{\sqrt{\delta}}$, we write the stationary DNLS equation:

$$(1 - |W_n|^2) W_n = \delta \left[W_{n+1} + W_{n-1} + i\gamma(-1)^n W_n \right], \quad 1 \leq n \leq N,$$

subject to the boundary condition $W_0 = 0$ and $W_{N+1} = \bar{W}_N$.

For $N = 1$, we have

$$(1 - |W_1|^2) W_1 = \delta \left[\bar{W}_1 - i\gamma W_1 \right].$$

Setting $W_1 = A_1^{1/2} e^{i\varphi_1}$, we obtain

$$A_1 = 1 - \delta \cos(2\varphi_1), \quad -\sin(2\varphi_1) - \gamma = 0,$$

which yields two branches by the two solutions of $\sin(2\varphi_1) = -\gamma$.

Stationary states of a quadrimer: $N = 2$

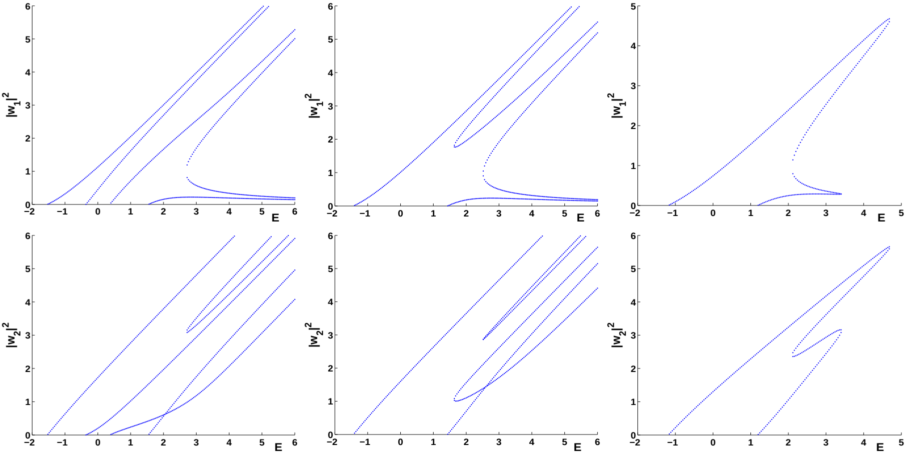

Figure: Nonlinear stationary states for $\gamma = 0.5$ (left), $\gamma = 0.75$ (middle), and $\gamma = 1.1$ (right). The top and bottom rows show components $|U_1|^2$ and $|U_2|^2$.

Spectral stability in PT-symmetric dNLS

Substitute $\mathbf{u}(t) = e^{-iEt}(\mathbf{U} + \mathbf{V}(t))$ to the PT-dNLS equation with $w_U = \bar{U}_{2N+1-n}$ and obtain the linearized time-evolution problem

$$i\frac{dV_n}{dt} + EV_n = V_{n+1} + V_{n-1} + i(-1)^n \gamma V_n + 2|U_n|^2 V_n + U_n^2 \bar{V}_n.$$

Then, use

$$\mathbf{V}(t) = \phi e^{-\lambda t} \quad \text{and} \quad \bar{\mathbf{V}}(t) = \psi e^{-\lambda t},$$

to obtain the spectral problem with the eigenvalue parameter λ

$$\left\{ \begin{array}{l} (E - i\lambda)\phi_n = \phi_{n+1} + \phi_{n-1} + i(-1)^n \gamma \phi_n + 2|U_n|^2 \phi_n + U_n^2 \psi_n, \\ (E + i\lambda)\psi_n = \psi_{n+1} + \psi_{n-1} - i(-1)^n \gamma \psi_n + \bar{U}_n^2 \phi_n + 2|U_n|^2 \psi_n. \end{array} \right.$$

Unless λ is real, ϕ_n and ψ_n are not complex-conjugate to each other.

Stability Theorem

Theorem

Consider the 2^N stationary solutions on S_N in the limit $E \to \infty$. There exists exactly one spectrally stable stationary state for sufficiently large E.

Using the rescaling $E = 1/\delta$, $\mathbf{U} = \mathbf{W}/\delta^{1/2}$, and $\lambda = \Lambda/\delta$, we obtain

$$\left\{ \begin{array}{l} (1-2|W_n|^2)\phi_n - W_n^2\psi_n = i\Lambda\phi_n + \delta\left(\phi_{n+1} + \phi_{n-1} + i(-1)^n\gamma\phi_n\right), \\ -\bar{W}_n^2\phi_n + (1-2|W_n|^2)\psi_n = -i\Lambda\psi_n + \delta\left(\psi_{n+1} + \psi_{n-1} - i(-1)^n\gamma\psi_n\right). \end{array} \right.$$

where we recall $W_n = e^{i\theta_n}(1 + \mathcal{O}(\delta))$.

For $\delta = 0$, there exists only one eigenvalue $\Lambda = 0$ of algebraic multiplicity $4N$. The multiple zero eigenvalue splits in $\mathbb{C}$ if $\delta \neq 0$.

The only spectrally stable stationary solution has the out-of-phase configuration for all phase differences in the sequence $\theta_{n+1} - \theta_n$.

Stationary states in the generalized PT-dNLS network

Consider the generalized PT-symmetric dNLS network as

$$\left\{ \begin{array}{l} i\dot{u}_n + v_n + i\gamma u_n + u_{n-1} + u_{n+1} = (|u_n|^2 + |v_n|^2)u_n, \\ i\dot{v}_n + u_n - i\gamma v_n + v_{n-1} + v_{n+1} = (|u_n|^2 + |v_n|^2)v_n, \end{array} \right. \qquad n \in S.$$

For every $\gamma \in (-1, 1)$, all solutions of the generalized dimer remain bounded for all times. All stationary solutions are also identical to those in the standard PT-NLS network.

Theorem

Consider the 2^N stationary solutions on S_N in the limit $E \to \infty$. There exists exactly two spectrally stable stationary states for sufficiently large E.

Stability of stationary states of a quadrimer: $N = 2$

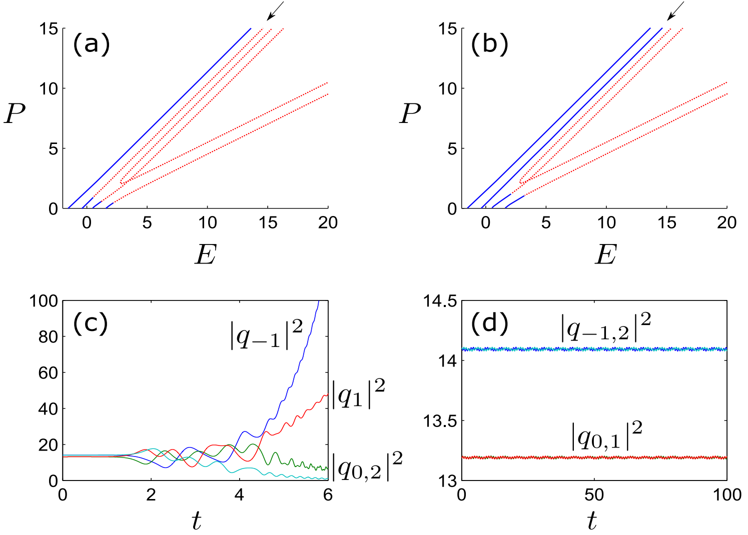

Figure: Left: the standard PT-dNLS equation. Right: the generalized PT-dNLS.

Another generalized PT-symmetric dNLS network

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$$\left\{ \begin{array}{l} i\dot{u}_n + v_n + i\gamma u_n + u_{n-1} + u_{n+1} = (|u_n|^2 + 2|v_n|^2)u_n + u_n^2 \bar{v}_n, \\ i\dot{v}_n + u_n - i\gamma v_n + v_{n-1} + v_{n+1} = (2|u_n|^2 + |v_n|^2)v_n + v_n^2 \bar{u}_n, \end{array} \right. \quad n \in S.$$

For every $\gamma \in \mathbb{R}$, all solutions of the generalized dimer remain bounded for all times $t \in \mathbb{R}$.

Theorem

For any $\gamma \in \mathbb{R}$, the stationary PT-dNLS equation on S_N admits exactly one PT-symmetric solution (unique up to a gauge transformation) such that

$$\left| |U_0|^2 - E \right| \leq C \quad \text{and} \quad |U_n|^2 \leq C E^{-1} \quad \text{for each } n \neq 0.$$

The stationary state is spectrally stable if $\gamma \in (-1, 1)$.

Open problems

- Global bounds on the solutions in the generalized PT-symmetric dNLS-type networks.

- Spectral stability of general nonlinear states in the generalized PT-symmetric dNLS-type networks.

- Nonlinear stability of spectrally stable stationary states.

- Extensions to multi-dimensional settings of the PT-dNLS equation.

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