Instability of peaked traveling waves in the Camassa–Holm models

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Section 1

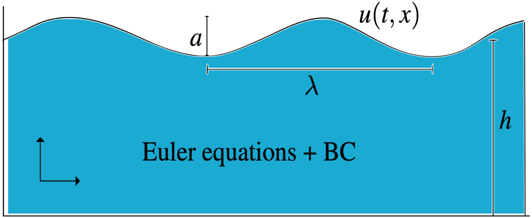
Camassa-Holm models

The Camassa-Holm equation

$$u_t - u_{txx} + 3\,u\,u_x = 2\,u_x u_{xx} + u\,u_{xxx} \qquad \text{(CH)}$$

models the propagation of unidirectional shallow water waves, where $u = u(t,x)$ represents the horizontal velocity at the free surface.

[Camassa & Holm, 1993] [Johnson, 2000] [Constantin & Lannes, 2009]

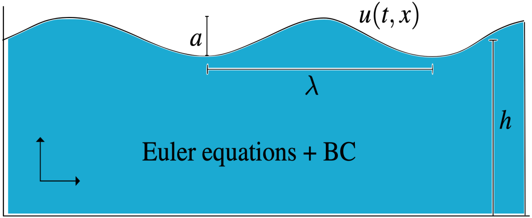


It was extended as the Degasperis–Procesi equation

$$u_t - u_{txx} + 4\,u\,u_x = 3\,u_x u_{xx} + u\,u_{xxx} \tag{DP}$$

at the same asymptotic accuracy.

[Degasperis & Procesi, 1999] [Constantin & Lannes, 2009]



It was further extended as the b-Camassa–Holm equation

$$u_t - u_{txx} + (b+1)\, u\, u_x = b\, u_x u_{xx} + u\, u_{xxx} \qquad \text{(b-CH)}$$

by using transformations of integrable KdV equation
[Dullin, Gottwald, & Holm, 2001] [Degasperis, Holm & Hone, 2002]

- ▷ CH and DP cases are integrable for $b = 2$ and $b = 3$.
- ▷ BBM equation for slowly varying waves:

$$u_t - u_{txx} + (b+1)\, u\, u_x = 0$$

- ▷ Admits both smooth and peaked traveling waves.
- ▷ Purely quadratic in the evolution form:

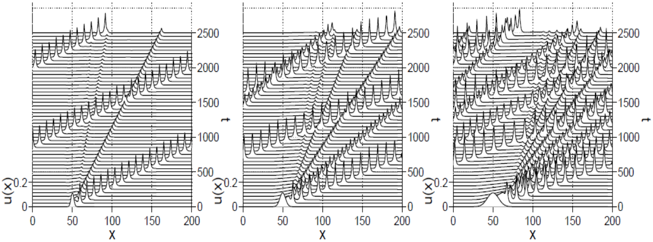
$$u_t = (1 - \partial_x^2)^{-1} \left[b\, u_x u_{xx} + u\, u_{xxx} - (b+1) u u_x \right].$$

Solitary waves in b-CH model

Similations of the b-family of Camassa-Holm equations

$$u_t - u_{txx} + (b+1)\, u\, u_x = b\, u_x u_{xx} + u\, u_{xxx}$$

starting with Gaussian initial data $u(0,x)$ [Holm & Staley, 2003]



Peaked solitary waves (*peakons*) are observed for $b > 1$

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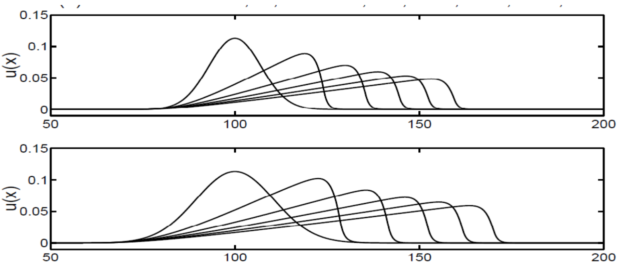

Rarefactive waves are observed for $b \in (-1, 1)$

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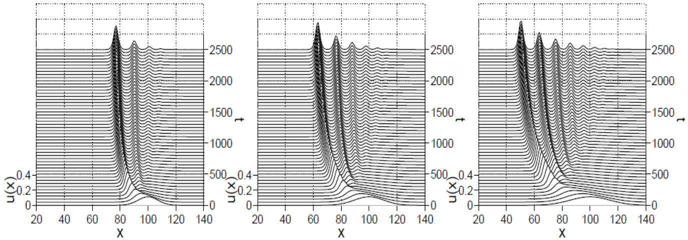

Smooth solitary waves (*leftons*) are observed for $b < -1$

Stability of solitary waves: state of the art

For traveling solitary waves satisfying $u(x) \to 0$ as $|x| \to \infty$

- Orbital stability of peakons in energy space
  $b = 2$: [Constantin & Strauss, 2000] [Constantin & Molinet, 2001]
  $b = 3$: [Lin & Liu, 2009]

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For solitary waves satisfying $u(x) \to k$ as $|x| \to \infty$ with $k > 0$:

- Orbital stability of smooth solitary waves in energy space
  $b = 2$: [Constantin & Strauss, 2002]
  $b = 3$: [Li & Liu & Wu, 2020]

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  $b = 3$: [Li & Liu & Wu, 2020]

Similar studies were developed for travelling periodic waves (smooth or peaked) in the CH equation ($b = 2$) [Lenells, 2004-2006]

Stability of solitary waves: new results

- Linear and nonlinear instability of peakons in $H^1 \cap W^{1,\infty}$
  $b = 2$: [Natali & P., 2020] [Madiyeva & P., 2021]

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- Spectral stability of smooth periodic waves in $L^2_{\rm per}$
  $b = 2$ [Geyer, Martins, Natali, & P., 2022]
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  $b > 1$ [Ehrman & Johnson, 2023]

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  $b = 2$ [Geyer, Martins, Natali, & P., 2022]
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Similar studies were developed for the cubic CH (Novikov) equation
[Chen & P., 2021], [Lafortune, 2023]

Section 2

Properties of the b-Camassa–Holm equation

Properties of the Camassa-Holm equation on the line

The local differential equation

$$u_t - u_{txx} + (b+1)\, u\, u_x = b\, u_x u_{xx} + u\, u_{xxx}$$

can be rewritten in the integral form of the perturbed Burgers equation

$$u_t + uu_x + \frac{1}{4}\varphi' * \left(bu^2 + (3-b)u_x^2\right) = 0,$$

where $\varphi := 2(1-\partial_x^2)^{-1}\delta = e^{-|x|}$ is the Green function.

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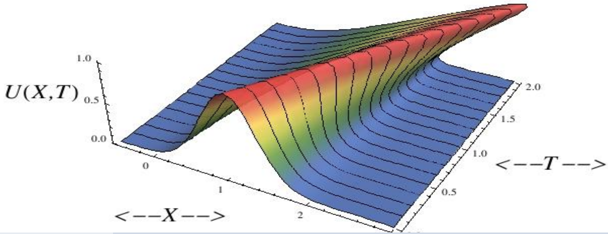

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We say that the dynamics leads to the wave breaking if

$$\|u(t,\cdot)\|_{L^\infty} < \infty, \quad \|u_x(t,\cdot)\|_{L^\infty} \to \infty \quad \text{as} \;\; t \to T < \infty$$

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Solutions of the Burgers equation $v_t + v v_x = 0$ with $v(0,x) = f(x)$ admit wave breaking if $f \in H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$:

$$v(t,x) = f(x - t v(t,x)) \quad \Rightarrow \quad v_x = \frac{f'(x-tv)}{1 + t f'(x-tv)}.$$

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The CH equation ($b = 2$) ...

- ▷ is locally well-posed in H^s, $s > 3/2$ [Escher & Yin, 2008; Zhou, 2010]
- ▷ has no continuous dependence in H^s, $s \leq 3/2$
  [Himonas, Grayshan, Holliman (2016)] [Guo, Liu, Molinet, Yin (2018)]
- ▷ is locally well-posed in $H^1 \cap W^{1,\infty}$.
  [De Lellis, Kappeler, Topalov (2007)] [Linares, Ponce, Sideris (2019)]

Hamiltonian structure of the b-CH equations

For $b = 2$, the Camassa–Holm equation

$$u_t - u_{txx} + 3\,u\,u_x = 2\,u_x u_{xx} + u\,u_{xxx}$$

has the first three conserved quantities

$$M(u) = \int u dx, \quad E(u) = \frac{1}{2}\int (u^2 + u_x^2) dx, \quad F(u) = \frac{1}{2}\int (u^3 + u u_x^2)\, dx.$$

(CH) can be written in Hamiltonian form in three ways:

$$\begin{aligned} u_t &= J F'(u), & J &= -(1-\partial_x^2)^{-1}\partial_x, \\ m_t &= J_m E'(m), & J_m &= -\left(m\partial_x + \partial_x m\right), \\ m_t &= J_m M'(m), & J_m &= -(2m\partial_x + m_x)(1-\partial_x^2)^{-1}\partial_x^{-1}(2\partial_x m - m_x). \end{aligned}$$

where $m = u - u_{xx}$.

Hamiltonian structure of the b-CH equations

For $b = 3$, the Degasperis–Procesi equation

$$u_t - u_{txx} + 4\,u\,u_x = 3\,u_x u_{xx} + u\,u_{xxx}$$

has the first three conserved quantities

$$M(u) = \int u dx, \quad E(u) = \frac{1}{2} \int u(1-\partial_x^2)(4-\partial_x^2)^{-1} u dx, \quad F(u) = \frac{1}{6} \int u^3 dx.$$

(DH) can be written in Hamiltonian form in two ways:

$$u_t = JF'(u), \qquad J = -(1-\partial_x^2)^{-1}(4-\partial_x^2)\partial_x,$$

$$m_t = J_m M'(m), \qquad J_m = -\frac{1}{2}(3m\partial_x + m_x)(1-\partial_x^2)^{-1}\partial_x^{-1}(3\partial_x m - m_x).$$

where $m = u - u_{xx}$.

Hamiltonian structure of the b-CH equations

For general $b \neq 1$, the b-Camassa–Holm equation

$$u_t - u_{txx} + (b+1)\, u\, u_x = b\, u_x u_{xx} + u\, u_{xxx}$$

can be written in Hamiltonian form:

$$m_t = J_m M'(m), \quad J_m := -\frac{1}{b-1}(bm\partial_x + m_x)(1-\partial_x^2)^{-1}\partial_x^{-1}(b\partial_x m - m_x).$$

where $m = u - u_{xx}$.

Section 3

Instability of peakons for $b = 2$

Standard approach to orbital stability of traveling waves

- Construct a linear combination of conserved quantities $\Lambda(u)$ such that the traveling wave ϕ is a critical point of Λ: $\underbrace{\Lambda'(\phi) = 0}_{\text{TW-eq}}$

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- If $\mathcal{L}$ has only one negative simple eigenvalue and a simple zero eigenvalue, then prove that the traveling wave ϕ is a constrained minimizer of energy, i.e. $\mathcal{L}|_{X_0} \geq 0$, where $X_0 \subset L^2$ is due to constraints

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- The traveling wave ϕ is orbitally stable in energy space if local well-posedness has been proven in the energy space.

Existence of peakons

Peakons exist in the weak form in $H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ for every $b \in \mathbb{R}$:

$$u(t,x) = ce^{-|x-ct|} = c\varphi(x-ct).$$

We can set $c = 1$ due to the scaling transformation.

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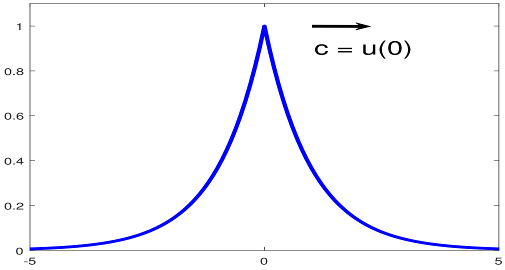

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We can set $c = 1$ due to the scaling transformation.

By using the traveling wave reduction $u(t,x) = \varphi(x-t)$ in

$$u_t + uu_x + \frac{1}{4}\varphi' * \left(bu^2 + (3-b)u_x^2\right) = 0$$

and integration once yields the integral equation

$$-\varphi + \frac{1}{2}\varphi^2 + \frac{1}{4}\varphi * \left(b\varphi^2 + (3-b)(\varphi')^2\right) = 0,$$
$$\Rightarrow -\varphi + \frac{1}{2}\varphi^2 + \frac{3}{4}\varphi * \varphi^2 = 0,$$

which is satisfied by $\varphi(x) = e^{-|x|}$.

Orbital stability of peakons in $H^1(\mathbb{R})$ for $b = 2$

Theorem (Constantin–Molinet (2001))

φ is a unique (up to translation) minimizer of $F(u)$ in $H^1(\mathbb{R})$ subject to fixed $E(u)$, where $F(u)$ and $E(u)$ are two conserved energies:

$$E(u) = \frac{1}{2}\int (u^2 + u_x^2)dx, \qquad F(u) = \frac{1}{2}\int (u^3 + uu_x^2)\, dx.$$

Theorem (Constantin–Strauss (2000))

For every small $\varepsilon > 0$, if the initial data satisfies

$$\|u_0 - \varphi\|_{H^1} < \left(\frac{\varepsilon}{3}\right)^4,$$

then the solution satisfies

$$\|u(t,\cdot) - \varphi(\cdot - \xi(t))\|_{H^1} < \varepsilon, \quad t \in (0,T),$$

where $\xi(t)$ is a point of maximum for $u(t,\cdot)$.

Instability of peakons in $H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ for $b = 2$

Consider solutions of the Cauchy problem:

$$\left\{ \begin{array}{l} u_t + uu_x + Q[u] = 0, \\ u|_{t=0} = u_0 \in H^1 \cap W^{1,\infty}, \end{array} \right. \qquad Q[u] := \frac{1}{4}\varphi' * \left(u^2 + \frac{1}{2}u_x^2 \right).$$

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Theorem (Natali–P. (2020))

For every $\delta > 0$, there exist $t_0 > 0$ and $u_0 \in H^1 \cap W^{1,\infty}$ satisfying

$$\|u_0 - \varphi\|_{H^1} + \|u_0' - \varphi'\|_{L^\infty} < \delta,$$

s.t. the unique solution $u \in C([0,T), H^1 \cap W^{1,\infty})$ with $T > t_0$ satisfies

$$\|u_x(t_0,\cdot) - \varphi'(\cdot - \xi(t_0))\|_{L^\infty} > 1,$$

where $\xi(t)$ is a point of peak of $u(t,\cdot)$ for $t \in [0,T)$.

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$Q[u]$ behaves better than uu_x:

- If $u \in H^1(\mathbb{R})$, then $Q[u] \in H^1(\mathbb{R})$ and hence continuous.
- If $u \in H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, then $Q[u]$ is Lipschitz continuous.
- If $u \in H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, method of characteristics can be used to analyze dynamics of the perturbed Burgers equation.

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One important property for continuous solutions with peaked corners:

If $u(t,\cdot) \in H^1(\mathbb{R}) \cap C^1(\mathbb{R}\backslash\{\xi(t)\})$ for $t \in [0,T)$, then $\xi(t) \in C^1(0,T)$ and

$$\frac{d\xi}{dt} = u(t,\xi(t)), \quad t \in (0,T).$$

For the peaked traveling wave $u(t,x) = u(x-ct)$,
this gives $c = u(0) := \max_{x\in\mathbb{R}} u(x)$.

Instability of peakons in $H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ for $b = 2$

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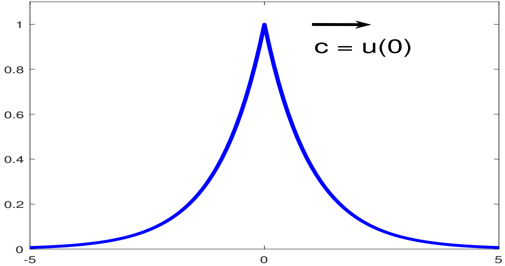
Here is a peaked solitary wave with a single peak:



Decomposition near a single peakon

Consider a decomposition:

$$u(t,x) = \varphi(x - t - a(t)) + v(t, x - t - a(t)), \quad t \in [0,T), \quad x \in \mathbb{R},$$

with the peak at $\xi(t) = t + a(t)$ for $v(t,\cdot) \in H^1(\mathbb{R}) \cap C^1(\mathbb{R}\backslash\{\xi(t)\})$.

Then, $a'(t) = v(t,0)$ and

$$v_t = (1-\varphi)v_x + (v|_{x=0} - v)\varphi' + (v|_{x=0} - v)v_x - \varphi' * (\varphi v + \frac{1}{2}\varphi' v_x) - Q[v].$$

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Due to

$$[v(0) - v(x)]\varphi'(x) - \varphi' * \varphi v - \frac{1}{2}\varphi' * \varphi' v_x = \varphi(x) \int_0^x v(y)dy,$$

the evolution of $v(t,x)$ simplifies to

$$v_t = (1-\varphi)v_x + \varphi \int_0^x v(t,y)dy + (v|_{x=0} - v)v_x - Q[v].$$

Nonlinear evolution

For the evolution problem:

$$\begin{cases} v_t = (1-\varphi)v_x + \varphi \int_0^x v(t,y)dy + (v|_{x=0} - v)v_x - Q[v], & t \in (0,T), \\ v|_{t=0} = v_0(x), \end{cases}$$

we can look for solutions with the method of characteristic curves:

$$x = X(t,s), \qquad v(t,X(t,s)) = V(t,s).$$

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$$\begin{cases} v_t = (1-\varphi)v_x + \varphi \int_0^x v(t,y)dy + (v|_{x=0} - v)v_x - Q[v], & t \in (0,T), \\ v|_{t=0} = v_0(x), \end{cases}$$

we can look for solutions with the method of characteristic curves:

$$x = X(t,s), \qquad v(t, X(t,s)) = V(t,s).$$

The characteristic coordinates $X(t,s)$ satisfies

$$\begin{cases} \frac{dX}{dt} = \varphi(X) - 1 + v(t,X) - v(t,0), & t \in (0,T), \\ X|_{t=0} = s. \end{cases}$$

Since φ and $v(t,\cdot)$ are Lipschitz for the solution in $H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, there exists the unique characteristic function $X(t,s)$ for each $s \in \mathbb{R}$. The peak location $X(t,0) = 0$ is invariant in time.

Nonlinear evolution

For the evolution problem:

$$\begin{cases} v_t = (1-\varphi)v_x + \varphi \int_0^x v(t,y)dy + (v|_{x=0} - v)v_x - Q[v], \quad t \in (0,T), \\ v|_{t=0} = v_0(x), \end{cases}$$

we can look for solutions with the method of characteristic curves:

$$x = X(t,s), \qquad v(t, X(t,s)) = V(t,s).$$

From the right side of the peak, $V_0(t) = v(t,0)$, $W_0(t) = v_x(t,0^+)$:

$$\frac{dW_0}{dt} = W_0 + V_0 + V_0^2 - \frac{1}{2}W_0^2 - P[v](0), \quad P[v] := \varphi * \left(v^2 + \frac{1}{2}v_x^2 \right).$$

We will show that $W_0(t)$ grows and may diverge in a finite time.

Proof of the nonlinear instability

From the orbital stability in $H^1(\mathbb{R})$ [A. Constantin, W. Strauss (2000)]
If $\|v_0\|_{H^1} < (\varepsilon/3)^4$, then

$$|V_0(t)| \leq \|v(t,\cdot)\|_{L^\infty} \leq \frac{1}{\sqrt{2}}\|v(t,\cdot)\|_{H^1} < \varepsilon.$$

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To show instability, we use eq. on the right side of the peak:

$$\frac{dW_0}{dt} = W_0 + V_0 + V_0^2 - \frac{1}{2}W_0^2 - P[v](0)$$

and since $P[v] > 0$, we have

$$\frac{dW_0}{dt} \leq W_0 + C\varepsilon \quad \Rightarrow \quad W_0(t) \leq \left[W_0(0) + C\varepsilon\right] e^t$$

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If $W_0(0) = -2C\varepsilon$, then

$$W_0(t) \leq -C\varepsilon e^t,$$

hence $|W_0(t_0)| \geq 1$ for $t_0 := -\log(C\varepsilon)$.

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The initial constraint $\|v_0\|_{L^\infty} + \|v_0'\|_{L^\infty} < \delta$, is satisfied
if $\forall \delta > 0$, $\exists \varepsilon > 0$ such that

$$\left(\frac{\varepsilon}{3}\right)^4 + 2C\varepsilon < \delta.$$

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$$|V_0(t)| \leq \|v(t,\cdot)\|_{L^\infty} \leq \frac{1}{\sqrt{2}}\|v(t,\cdot)\|_{H^1} < \varepsilon.$$

To show the finite-time wave breaking, we estimate

$$\frac{dW_0}{dt} = W_0 + V_0 + V_0^2 - \frac{1}{2}W_0^2 - P[v](0) \leq W_0 - \frac{1}{2}W_0^2 + C\varepsilon.$$

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$$|V_0(t)| \leq \|v(t,\cdot)\|_{L^\infty} \leq \frac{1}{\sqrt{2}}\|v(t,\cdot)\|_{H^1} < \varepsilon.$$

By the ODE comparison theory, $W_0(t) \leq \overline{W}(t)$, where the supersolution satisfies

$$\frac{d\overline{W}}{dt} = \overline{W} - \frac{1}{2}\overline{W}^2 + C\varepsilon$$

with $W_0(0) = \overline{W}(0) = -C\varepsilon$ and $\overline{W}(t) \to -\infty$ as $t \to \overline{T}$.

Illustration of the peakon instability (periodic case)

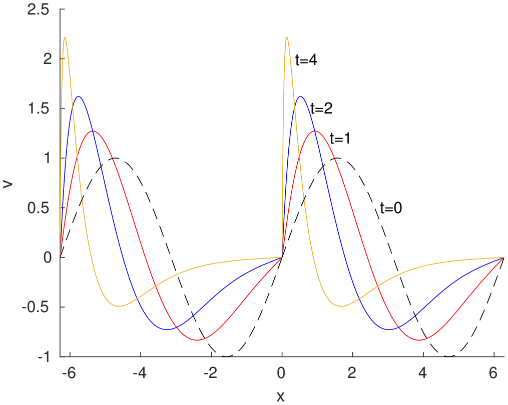

Figure: The plots of perturbation $v(t,x)$ to the peaked wave versus x on $[-2\pi, 2\pi]$ for different values of t in the case $v_0(x) = \sin(x)$.

Section 4

Spectral instability of peakons for every $b \in \mathbb{R}$

Linearized equation for every $b \in \mathbb{R}$

Truncation of the quadratic terms yields the linearized problem for perturbations in $H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$:

$$v_t = (1-\varphi)v_x + (b-2)(v|_{x=0} - v)\varphi' + \frac{1}{2}(b-3)\varphi * (\varphi' v) - \frac{1}{2}(2b-3)\varphi' * (\varphi v),$$

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Question: Can we predict instability of peakons from analysis of the associated linearized operator in $L^2(\mathbb{R})$?

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The linearized operator is

$$L = (1-\varphi)\partial_x - (b-2)\varphi' + K,$$

where $K : L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ is a compact (Hilbert–Schmidt) operator. Since $\varphi \in H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, the natural domain of L in $L^2(\mathbb{R})$ is

$$\mathrm{Dom}(L) = \left\{ v \in L^2(\mathbb{R}) : \quad (1-\varphi)v' \in L^2(\mathbb{R}) \right\}.$$

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$H^1(\mathbb{R})$ is continuously embedded into $\text{Dom}(L)$. However, it is not equivalent to $\text{Dom}(L)$ because $\varphi' \in \text{Dom}(L)$ but $\varphi' \notin H^1(\mathbb{R})$.

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Question: How can we redefine L from $H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$ to $\text{Dom}(L) \subset L^2(\mathbb{R})$ to study spectral stability of peakons?

Answering of these questions

It can be checked directly that

$$L\varphi = (2-b)\varphi' \text{ and } L\varphi' = 0.$$

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Starting with $v \in H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$, we write

$$v = v|_{x=0}\varphi + \tilde{v} \quad \text{such that } \ \tilde{v}(t,0) = 0.$$

Then,

$$v_t = Lv + (b-2)v|_{x=0}\varphi' \quad \Rightarrow \quad \tilde{v}_t = L\tilde{v} - \frac{3}{2}(b-2)\langle \varphi\varphi', \tilde{v}\rangle \varphi$$

Linear evolution is now well-defined for $\tilde{v} \in \text{Dom}(L) \subset L^2(\mathbb{R})$ for which $\tilde{v}(t,0)$ may not exist.

Answering of these questions

It can be checked directly that

$$L\varphi = (2-b)\varphi' \text{ and } L\varphi' = 0.$$

Moreover, we can use the secondary decomposition

$$\tilde{v}(t,x) = \alpha(t)\varphi(x) + \beta(t)\varphi'(x) + w(t,x)$$

and obtain the homogeneous equation $w_t = Lw$ and

$$\frac{d\alpha}{dt} = (2-b)\beta + \frac{3}{2}(2-b)\langle \phi\phi', w\rangle, \quad \frac{d\beta}{dt} = (2-b)\alpha.$$

For $b \neq 2$, we have instability of peakons in $\mathrm{Dom}(L)$ with $w = 0$. For $b = 2$, we have to analyze the spectrum of L in $L^2(\mathbb{R})$.

Spectrum of a linear operator

Let A be a linear operator on a Banach space X with $\mathrm{Dom}(A) \subset X$. The complex plane $\mathbb{C}$ is decomposed into the resolvent set $\rho(A)$ and the spectrum $\sigma(A) = \mathbb{C} \setminus \rho(A)$, the latter consists of the following three disjoint sets:

1. the point spectrum
$$\sigma_{\mathrm{p}}(A) = \{\lambda : \quad \mathrm{Ker}(A - \lambda I) \neq \{0\}\},$$
2. the residual spectrum
$$\sigma_{\mathrm{r}}(A) = \{\lambda : \quad \mathrm{Ker}(A - \lambda I) = \{0\}, \quad \mathrm{Ran}(A - \lambda I) \neq X\},$$
3. the continuous spectrum
$$\sigma_{\mathrm{c}}(A) = \{\lambda : \quad \mathrm{Ker}(A - \lambda I) = \{0\}, \quad \mathrm{Ran}(A - \lambda I) = X, \\ (A - \lambda I)^{-1} : X \to X \text{ is unbounded}\}.$$

Spectrum of a linear operator

Theorem (Lafortune–P (2022))

The spectrum of L with $\mathrm{Dom}(L) \subset L^2(\mathbb{R})$

$$\sigma(L) = \left\{ \lambda \in \mathbb{C} : \quad |\mathrm{Re}(\lambda)| \leq \left| \frac{5}{2} - b \right| \right\}.$$

Moreover,

- $\sigma_p(L)$ *is located for* $0 < |\mathrm{Re}(\lambda)| < \frac{5}{2} - b$ *if* $b < \frac{5}{2}$
- $\sigma_r(L)$ *is located for* $0 < |\mathrm{Re}(\lambda)| < b - \frac{5}{2}$ *if* $b > \frac{5}{2}$
- $\sigma_c(L)$ *is located for* $\mathrm{Re}(\lambda) = 0$ *and* $\mathrm{Re}(\lambda) = \pm \left| \frac{5}{2} - b \right|$
- $\lambda = 0$ *is the embedded eigenvalue for every* b*.*

$\Rightarrow$ the peakon is linearly unstable for perturbations in $\mathrm{Dom}(L)$ for every $b \neq \frac{5}{2}$.

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CH and DP have different types of peakon instability

$b = 2$: $\|v(t,\cdot)\|_{L^2(-\infty,0)}$ grows due to point spectrum

$b = 3$: $\|v(t,\cdot)\|_{L^2(0,\infty)}$ grows due to residual spectrum

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Instability in the vertical strip holds for peaked waves in the reduced Ostrovsky equation $u_t + uu_x = \partial_x^{-1} u$ [Geyer & P. (2020)] and for Euler flows [Shvidkoy & Latushkin (2003)]

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For fixed b, the width of the instability strip changes if L is considered in $\mathrm{Dom}(L) \subset H^s(\mathbb{R})$ with $s \neq 0$. [Lafortune (2023)].

How do we obtain this result?

Recall that $L = L_0 + K$, where $L_0 := (1 - \varphi)\partial_x - (b - 2)\varphi'$ with

$$\mathrm{Dom}(L) = \mathrm{Dom}(L_0) = \left\{ v \in L^2(\mathbb{R}) : \quad (1 - \varphi)v' \in L^2(\mathbb{R}) \right\}$$

and $K : L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ is a compact (Hilbert–Schmidt) operator.

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Theorem (Geyer & P (2020))

Let $L : \mathrm{Dom}(L) \subset X \to X$ *and* $L_0 : \mathrm{Dom}(L_0) \subset X \to X$ *be linear operators on Hilbert space X with the same domain such that* $L - L_0 = K$ *is a compact operator in X. Assume that the intersections* $\sigma_{\mathrm{p}}(L) \cap \rho(L_0)$ *and* $\sigma_{\mathrm{p}}(L_0) \cap \rho(L)$ *are empty. Then,* $\sigma(L) = \sigma(L_0)$.

How do we obtain this result?

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and $K : L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ is a compact (Hilbert–Schmidt) operator.

Theorem (Bühler & Salamon (2018))

Let $L : \mathrm{Dom}(L) \subset X \to X$ *be a linear operator on Hilbert space X and* $L^* : \mathrm{Dom}(L^*) \subset X \to X$ *be the adjoint operator. Assume that* $\sigma_{\mathrm{p}}(L)$ *is empty. Then,* $\sigma_{\mathrm{r}}(L) = \sigma_{\mathrm{p}}(L^*)$.

How do we obtain this result?

Recall that $L = L_0 + K$, where $L_0 := (1-\varphi)\partial_x - (b-2)\varphi'$ with

$$\text{Dom}(L) = \text{Dom}(L_0) = \left\{ v \in L^2(\mathbb{R}) : \quad (1-\varphi)v' \in L^2(\mathbb{R}) \right\}$$

and $K : L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ is a compact (Hilbert–Schmidt) operator.

Truncated equation $L_0 v = \lambda v$ is the first-order equation

$$(1-\varphi)\frac{dv}{dx} + (2-b)\varphi' v = \lambda v$$

with the exact solution

$$v(x) = \begin{cases} v_+ e^{\lambda x}(1-e^{-x})^{2+\lambda-b}, & x > 0, \\ v_- e^{\lambda x}(1-e^{x})^{2-\lambda-b}, & x < 0, \end{cases}$$

If $\text{Re}(\lambda) > 0$, then $v_+ = 0$ and $\text{Re}(\lambda) < \frac{5}{2} - b$.

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and $K : L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$ is a compact (Hilbert–Schmidt) operator.

Truncated equation $L_0^* v = \lambda v$ is the first-order equation

$$-(1-\varphi)\frac{dv}{dx} + (3-b)\varphi' v = \lambda v$$

with the exact solution

$$v(x) = \begin{cases} v_+ e^{-\lambda x}(1-e^{-x})^{b-3-\lambda}, & x > 0, \\ v_- e^{-\lambda x}(1-e^{x})^{b-3+\lambda}, & x < 0, \end{cases}$$

If $\mathrm{Re}(\lambda) > 0$, then $v_- = 0$ and $\mathrm{Re}(\lambda) < b - \frac{5}{2}$.

Summary

We have considered the *b*-Camassa–Holm equation

$$u_t - u_{txx} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx}$$

which models unidirectional small-amplitude shallow water waves.

- ▷ Peaked traveling waves are unstable in $H^1 \cap W^{1,\infty}$
  - ▷ LWP only holds in $H^1 \cap W^{1,\infty}$.
  - ▷ Perturbations are bounded in H^1 (at least for $b = 2$).
  - ▷ Perturbations grow in $W^{1,\infty}$ norm.
  - ▷ Spectral instability holds for every b.

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MANY THANKS FOR YOUR ATTENTION!