Nonlinear Schrödinger equation on a periodic graph

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Summary

NLS with Periodic Potentials

NLS on the periodic graph

Linear theory

Homogenization of the NLS equation

Stationary states on the periodic graph

Conclusion

Introduction: periodic potentials

Let us recall homogenization of the nonlinear Schrödinger equation

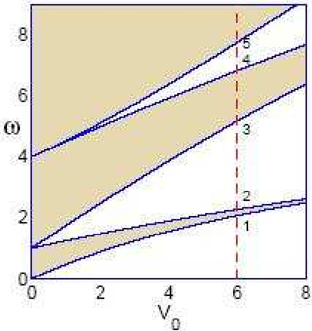
$$iu_t = -u_{xx} + V(x)u \pm |u|^2 u,$$

with a periodic potential, e.g. $V(x) = V_0 \sin^2(x)$.
D.P. *Localization in Periodic Potentials* (Cambridge University Press, 2011)

Stationary solutions $u(x,t) = \phi(x)e^{-i\omega t}$ with $\omega \in \mathbb{R}$ satisfy a stationary Schrödinger equation with a periodic potential

$$\omega\phi = -\phi_{xx} + V(x)\phi \pm |\phi|^2\phi$$

Spectrum of $L = -\partial_x^2 + V(x)$ for $V(x) = V_0 \sin^2(x)$ and $N = 1$:



Floquet–Bloch spectrum

The spectral problem with a bounded 2π-periodic potential V,

$$\omega W = -\partial_x^2 W + V(x)W, \quad x \in \mathbb{R},$$

has a purely continuous spectrum in $L^2(\mathbb{R})$. The spectrum can be found by using Bloch waves

$$W(x) = e^{i\ell x} f(\ell, x), \quad \ell,\ x \in \mathbb{R},$$

where $f(\ell, \cdot)$ satisfy the continuation conditions

$$f(\ell, x) = f(\ell, x + 2\pi), \quad f(\ell, x) = f(\ell + 1, x)e^{ix}, \quad \ell,\ x \in \mathbb{R}.$$

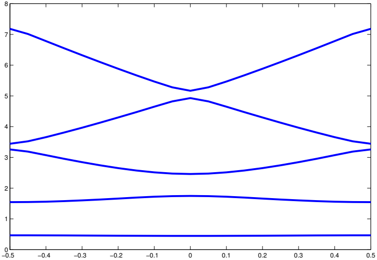
One can restrict the definition of $f(\ell, x)$ for $\ell \in \mathbb{T}_1 = \mathbb{R}/\mathbb{Z}$ and $x \in \mathbb{T}_{2\pi} = \mathbb{R}/(2\pi\mathbb{Z})$.

For a fixed $\ell \in \mathbb{T}_1$, the Bloch waves are found from the periodic spectral problem,

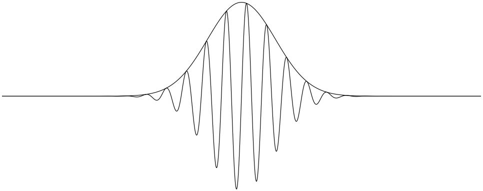
$$-(\partial_x + i\ell)^2 f + V(x)f = \omega(\ell)f, \quad x \in \mathbb{T}_{2\pi}.$$

There exists a Schauder basis $\{f^{(m)}(\ell, \cdot)\}_{m\in\mathbb{N}}$ in $L^2(0, 2\pi)$ for an increasing sequence of eigenvalues $\{\omega^{(m)}(\ell)\}_{m\in\mathbb{N}}$.

Modulated Bloch waves



Pick $m_0 \in \mathbb{N}$ and $\ell_0 \in \mathbb{T}_1$ such that $\omega^{(m)}(\ell_0) \neq \omega^{(m_0)}(\ell_0)$ for every $m \neq m_0$.



Homogenization of the NLS equation

The NLS equation with a bounded periodic potential V,

$$iu_t = -u_{xx} + V(x)u \pm |u|^2 u,$$

can be reduced to a homogeneous NLS equation

$$i\partial_T A = -\frac{1}{2}\partial_\ell^2 \omega^{(m_0)}(\ell_0)\partial_X^2 A \pm \nu |A|^2 A, \quad \nu = \frac{\|f^{(m_0)}(\ell_0,\cdot)\|_{L^4_{\rm per}}^4}{\|f^{(m_0)}(\ell_0,\cdot)\|_{L^2_{\rm per}}^2}$$

Theorem (Schneider–Uecker, 2006; Dohnal, 2008; Ilan–Weinstein, 2010)

Fix $m_0 \in \mathbb{N}$, $\ell_0 \in \mathbb{T}_1$, and assume $\omega^{(m)}(\ell_0) \neq \omega^{(m_0)}(\ell_0)$ for every $m \neq m_0$. Then, for every $C_0 > 0$ and $T_0 > 0$, there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all solutions $A \in C(\mathbb{R}, H^3(\mathbb{R}))$ of the homogeneous NLS equation with

$$\sup_{T\in[0,T_0]} \|A(T,\cdot)\|_{H^3} \leq C_0$$

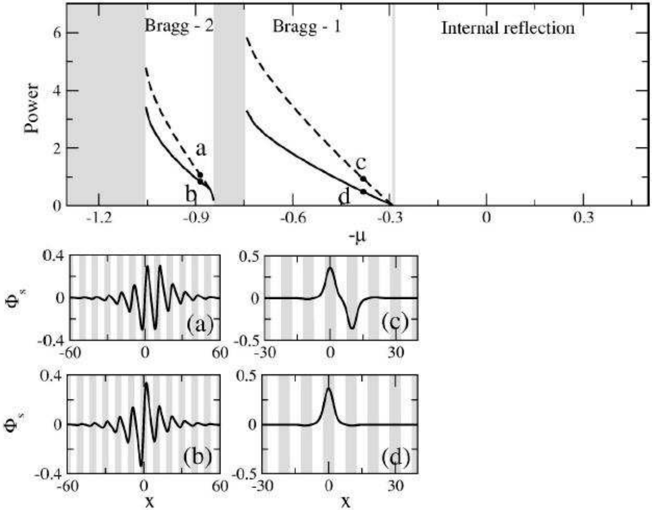
and for all $\varepsilon \in (0,\varepsilon_0)$, there are solutions $u \in C([0,T_0/\varepsilon^2], L^\infty(\mathbb{R}))$ of the periodic NLS equation satisfying the bound

$$\sup_{t\in[0,T_0/\varepsilon^2]} \sup_{x\in\mathbb{R}} \left| u(t,x) - \varepsilon A(\varepsilon^2 t, \varepsilon(x - c_{\rm gr}t)) f^{(m_0)}(\ell_0,x) e^{i\ell_0 x} e^{-i\omega^{(m_0)}(\ell_0)t} \right| \leq C\varepsilon^{3/2}.$$

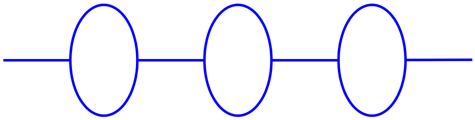
Application of the NLS equation to existence of nonlinear bound states

In the defocusing case, the nonlinear bound states bifurcate if $\partial_\ell^2 \omega^{(m_0)}(\ell_0) < 0$. In the focusing case, the nonlinear bound states bifurcate if $\partial_\ell^2 \omega^{(m_0)}(\ell_0) > 0$.

For $V(x) = V_0 \sin^2(x)$ and the defocusing case, the bifurcation diagram is



Periodic Graph



Consider the simplest periodic graph, where curvatures play no role:

$$\Gamma = \oplus_{n\in\mathbb{Z}}\Gamma_n, \qquad \text{with} \qquad \Gamma_n = \Gamma_{n,0} \oplus \Gamma_{n,+} \oplus \Gamma_{n,-},$$

where

$$\Gamma_{n,0} \quad \text{is identified with} \quad I_{n,0} = [2\pi n, 2\pi n + \pi]$$

and

$$\Gamma_{n,\pm} \quad \text{are identified with} \quad I_{n,\pm} = [2\pi n + \pi, 2\pi(n+1)]$$

Functions on graph

Wave functions $u : \Gamma \to \mathbb{C}$ are defined on the graph Γ in the pointwise sense:

$$u_{n,0} \quad \text{on} \quad I_{n,0} = [2\pi n, 2\pi n + \pi]$$

and

$$u_{n,\pm} \quad \text{on} \quad I_{n,\pm} = [2\pi n + \pi, 2\pi(n+1)]$$

subject to the Kirchhoff boundary conditions at the vertices.

$$\left\{ \begin{array}{l} u_{n,0}(2\pi n + \pi) = u_{n,+}(2\pi n + \pi) = u_{n,-}(2\pi n + \pi), \\ u_{n+1,0}(2\pi(n+1)) = u_{n,+}(2\pi(n+1)) = u_{n,-}(2\pi(n+1)), \end{array} \right.$$

and

$$\left\{ \begin{array}{l} \partial_x u_{n,0}(2\pi n + \pi) = \partial_x u_{n,+}(2\pi n + \pi) + \partial_x u_{n,-}(2\pi n + \pi), \\ \partial_x u_{n+1,0}(2\pi(n+1)) = \partial_x u_{n,+}(2\pi(n+1)) + \partial_x u_{n,-}(2\pi(n+1)). \end{array} \right.$$

The NLS equation on the periodic graph Γ

Collect all piecewise defined functions on the real line:

$$u_0(x) = \cup_{n\in\mathbb{Z}} \begin{cases} u_{n,0}(x), & x \in I_{n,0} := [2\pi n, 2\pi n + \pi], \\ 0, & \text{elsewhere}, \end{cases}$$

and

$$u_\pm(x) = \cup_{n\in\mathbb{Z}} \begin{cases} u_{n,\pm}(x), & x \in I_{n,\pm} := [2\pi n + \pi, 2\pi(n+1)), \\ 0, & \text{elsewhere}. \end{cases}$$

The three-component vector $U = (u_0,\ u_+, u_-)$: satisfies the NLS equation

$$i\partial_t U + \partial_x^2 U + |U|^2 U = 0, \quad t \in \mathbb{R}, \quad x \in \mathbb{R} \setminus \{k\pi : k \in \mathbb{Z}\},$$

subject to the Kirchhoff boundary conditions at the vertex points $\{k\pi : k \in \mathbb{Z}\}$.

Motivations

- Understand differences between the NLS with a bounded periodic potential and the NLS with vertex singularities due to the periodic graph Γ.
- Study homogenizations of the NLS equation on the periodic graph.
- Construct nonlinear bound states on the periodic graph.

References:

- S. Gilg, D.P., and G. Schneider, “Validity of the NLS approximation for periodic quantum graphs”, Nonlinear Differential Equations and Applications **23** (2016), 63 (30 pages).
- D.P. and G. Schneider, “Bifurcations of standing localized waves on periodic graphs”, Annales Henri Poincare **18** (2017), 1185–1211.

Linear spectral problem

The spectral problem on the periodic graph Γ:

$$\lambda w = -\partial_x^2 w, \quad x \in \Gamma,$$

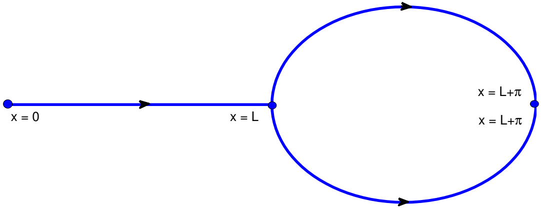
subject to the Kirchhoff boundary conditions for $n \in \mathbb{Z}$,

$$\begin{cases} w_{n,0}(2\pi n + \pi) = w_{n,+}(2\pi n + \pi) = w_{n,-}(2\pi n + \pi), \\ w_{n+1,0}(2\pi(n+1)) = w_{n,+}(2\pi(n+1)) = w_{n,-}(2\pi(n+1)), \end{cases}$$

and

$$\begin{cases} \partial_x w_{n,0}(2\pi n + \pi) = \partial_x w_{n,+}(2\pi n + \pi) + \partial_x w_{n,-}(2\pi n + \pi), \\ \partial_x w_{n+1,0}(2\pi(n+1)) = \partial_x w_{n,+}(2\pi(n+1)) + \partial_x w_{n,-}(2\pi(n+1)). \end{cases}$$

E. Korotyaev and I. Lobanov, Ann. Henri Poincare **8** (2007), 1151
P. Kuchment and O. Post, Commun Math. Phys. **275** (2007), 805



Decomposition of the spectrum on Γ

Lemma

The linear operator $-\partial_x^2 : \mathcal{D}(\Gamma) \to L^2(\Gamma)$ is self-adjoint with the domain $\mathcal{D}(\Gamma) \subset H^2(\Gamma)$. Its spectrum $\sigma(-\partial_x^2)$ is positive and consists of two parts.

Integrating by parts with Kirchhoff boundary conditions, we have

$$\lambda \|w\|^2_{L^2(\Gamma)} = \|\partial_x w\|^2_{L^2(\Gamma)} \geq 0.$$

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The first part of $\sigma(-\partial_x^2)$ corresponds to the eigenfunctions of the form

$$\left\{ \begin{array}{ll} w_{n,0}(x) = 0, & x \in [2\pi n, 2\pi n + \pi], \\ w_{n,+}(x) = -w_{n,-}(x), & x \in [2\pi n + \pi, 2\pi(n+1)], \end{array} \right. \quad n \in \mathbb{Z}.$$

Clearly, $\lambda = m^2$, $m \in \mathbb{N}$ is an eigenvalue of infinite multiplicity with the eigenfunction $w_{n,\pm}(x) = \pm\delta_{n,k} \sin[m(x - 2\pi n)]$, $k \in \mathbb{Z}$.

The second part of $\sigma(-\partial_x^2)$ corresponds to the eigenfunctions of the form

$$w_{n,+}(x) = w_{n,-}(x), \quad x \in [2\pi n + \pi, 2\pi(n+1)], \quad n \in \mathbb{Z}.$$

Construction of symmetric eigenfunctions

Let us parameterize the spectral parameter $\lambda = \omega^2$. Then, solutions of ODEs are found in terms of the boundary conditions:

$$\left\{ \begin{array}{ll} w_{n,0}(x) = a_n \cos(\omega(x - 2\pi n)) + b_n \sin(\omega(x - 2\pi n)), & x \in [2\pi n, 2\pi n + \pi], \\ w_{n,\pm}(x) = c_n \cos(\omega(x - 2\pi n - \pi)) + d_n \sin(\omega(x - 2\pi n - \pi)), & x \in [2\pi n + \pi, 2\pi(n + \end{array} \right.$$

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Kirchhoff boundary conditions yield

$$\begin{cases} c_n = a_n \cos(\omega\pi) + b_n \sin(\omega\pi), \\ 2d_n = -a_n \sin(\omega\pi) + b_n \cos(\omega\pi), \end{cases}$$

and

$$\begin{cases} a_{n+1} = c_n \cos(\omega\pi) + d_n \sin(\omega\pi), \\ b_{n+1} = -2c_n \sin(\omega\pi) + 2d_n \cos(\omega\pi). \end{cases}$$

The monodromy matrix

$$M(\omega) := \begin{bmatrix} \cos(\omega\pi) & \sin(\omega\pi) \\ -2\sin(\omega\pi) & 2\cos(\omega\pi) \end{bmatrix} \begin{bmatrix} \cos(\omega\pi) & \sin(\omega\pi) \\ -\frac{1}{2}\sin(\omega\pi) & \frac{1}{2}\cos(\omega\pi) \end{bmatrix}$$

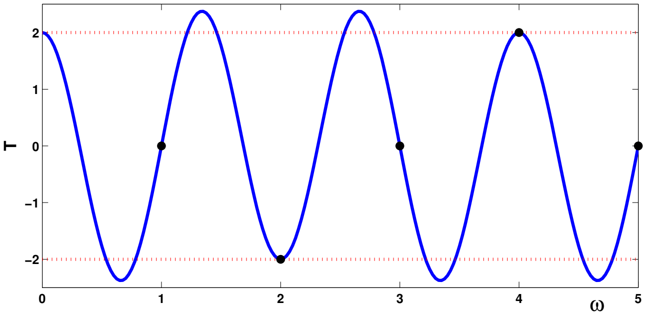
satisfies $\det(M) = 1$ and $\operatorname{tr}(M) = 2\cos(\omega\pi)^2 - \frac{5}{2}\sin(\omega\pi)^2$.

The symmetric part of the spectrum

Trace of the monodromy matrix:

$$T(\omega) = 2\cos(\omega\pi)^2 - \frac{5}{2}\sin(\omega\pi)^2 \in [-2, 2].$$

The spectrum $\sigma(-\partial_x^2)$ in $L^2(\Gamma)$ consists of eigenvalues $\{m^2\}_{m\in\mathbb{N}}$ of infinite multiplicity and a countable set of spectral bands $\{\sigma_k\}_{k\in\mathbb{N}}$.



Floquet–Bloch spectrum

We define the Bloch waves on the periodic graph Γ in the pointwise sense:

$$W(x) = e^{i\ell x} f(\ell, x), \quad \ell \in \mathbb{R}, \quad x \in \mathbb{R} \backslash \{\pi n, \; n \in \mathbb{N}\},$$

where $f(\ell, \cdot) = (f_0, f_+, f_-)(\ell, \cdot)$ is a 2π-periodic function for every $\ell \in \mathbb{R}$ satisfying the ℓ-dependent Kirchhoff boundary conditions

$$\left\{ \begin{array}{l} f_0(\ell, \pi) = f_+(\ell, \pi) = f_-(\ell, \pi), \\ f_0(\ell, 0) = f_+(\ell, 2\pi) = f_-(\ell, 2\pi) \end{array} \right.$$

and

$$\left\{ \begin{array}{l} (\partial_x + i\ell) f_0(\ell, \pi) = (\partial_x + i\ell) f_+(\ell, \pi) + (\partial_x + i\ell) f_-(\ell, \pi), \\ (\partial_x + i\ell) f_0(\ell, 0) = (\partial_x + i\ell) f_+(\ell, 2\pi) + (\partial_x + i\ell) f_-(\ell, 2\pi). \end{array} \right.$$

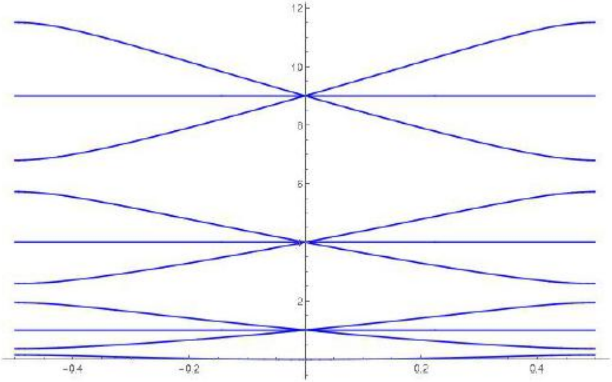
Note that $e^{i\ell x}$ is defined for $x \in \mathbb{R}$ but is not defined for $x \in \Gamma$.

Again, one can restrict the definition of $f(\ell, x)$ for $\ell \in \mathbb{T}_1 = \mathbb{R}/\mathbb{Z}$ and $x \in \mathbb{T}_{2\pi} = \mathbb{R}/(2\pi\mathbb{Z})$.

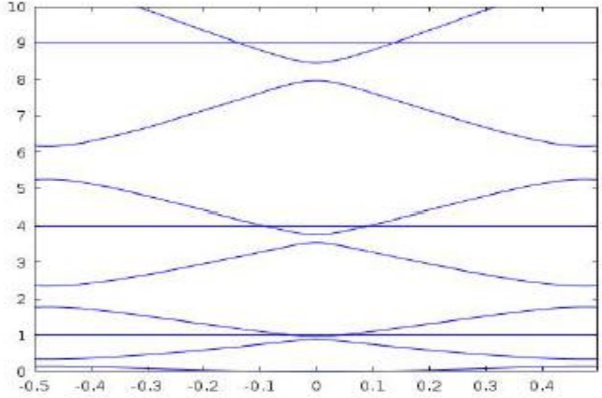
For a fixed $\ell \in \mathbb{T}_1$, the Bloch waves are found from the periodic spectral problem,

$$-(\partial_x + i\ell)^2 f = \omega(\ell) f, \quad x \in \mathbb{T}_{2\pi}.$$

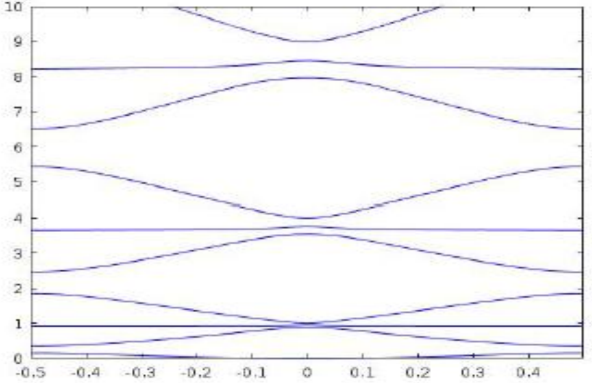
Numerical approximation of spectral bands: $L = \pi$



Numerical approximation of spectral bands: $L > \pi$



Numerical approximation of spectral bands: semi-rings of different lengths



The NLS equation on the periodic graph

The NLS equation on the periodic graph Γ written as the evolutionary problem for $U = (u_0,\ u_+, u_-)$:

$$i\partial_t U + \partial_x^2 U + |U|^2 U = 0, \quad t \in \mathbb{R}, \quad x \in \mathbb{R} \setminus \{k\pi : k \in \mathbb{Z}\},$$

subject to the Kirchhoff boundary conditions at the vertex points.

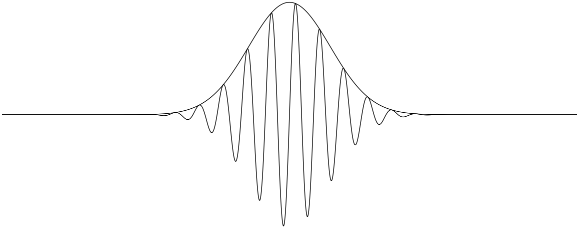


Figure: A schematic representation of the asymptotic solution to the NLS equation on the periodic graph Γ.

Homogeneous NLS equation

The asymptotic solution in the form

$$U(t,x) = \varepsilon A(T,X) f^{(m_0)}(\ell_0, x) e^{i\ell_0 x} e^{-i\omega^{(m_0)}(\ell_0)t} + \text{higher-order terms},$$

with $T = \varepsilon^2 t$ and $X = \varepsilon(x - c_g t)$ satisfies the homogeneous NLS equation

$$i\partial_T A + \frac{1}{2}\partial_\ell^2 \omega^{(m_0)}(\ell_0)\partial_X^2 A + \nu |A|^2 A = 0, \quad \nu = \frac{\|f^{(m_0)}(\ell_0,\cdot)\|_{L^4_{\text{per}}}^4}{\|f^{(m_0)}(\ell_0,\cdot)\|_{L^2_{\text{per}}}^2}.$$

Theorem (Gilg–Schneider-P, 2016)

Fix $m_0 \in \mathbb{N}$, $\ell_0 \in \mathbb{T}_1$, and assume $\omega^{(m)}(\ell_0) \neq \omega^{(m_0)}(\ell_0)$ for every $m \neq m_0$. Then, for every $C_0 > 0$ and $T_0 > 0$, there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all solutions $A \in C(\mathbb{R}, H^3(\mathbb{R}))$ of the homogeneous NLS equation with

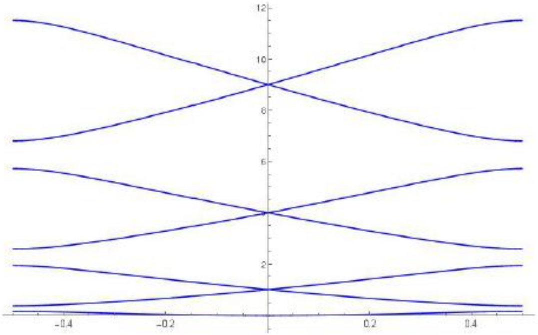
$$\sup_{T \in [0,T_0]} \|A(T,\cdot)\|_{H^3} \leq C_0$$

and for all $\varepsilon \in (0, \varepsilon_0)$, there are solutions $U \in C([0, T_0/\varepsilon^2], L^\infty(\mathbb{R}))$ to the NLS equation on the periodic graph Γ satisfying the bound

$$\sup_{t \in [0,T_0/\varepsilon^2]} \sup_{x \in \mathbb{R}} \left| U(t,x) - \varepsilon A(T,X) f^{(m_0)}(\ell_0,x) e^{i\ell_0 x} e^{-i\omega^{(m_0)}(\ell_0)t} \right| \leq C\varepsilon^{3/2}.$$

Extension to the Dirac equations

The symmetry constraints $u_{n,+}(t,x) = u_{n,-}(t,x)$ is invariant under the time evolution of the NLS equation on the periodic graph Γ. Under the constraints, the spectral bands feature Dirac points and no flat bands.



Homogeneous Dirac equations

The asymptotic solution in the form

$$U(t,x) = \varepsilon A_+(T,X) f^+(0,x) e^{-i\omega^+(0)t} + \varepsilon A_-(T,X) f^-(0,x) e^{-i\omega^-(0)t} + \text{higher-order terms},$$

with $T = \varepsilon^2 t$ and $X = \varepsilon^2 x$ satisfies the homogeneous Dirac equations

$$\begin{cases} i\partial_T A_+ + i\partial_\ell \omega^+(0)\partial_X A_+ + \sum_{j_1,j_2,j_3\in\{+,-\}} \nu^+_{j_1 j_2 j_3} A_{j_1} A_{j_2} \overline{A_{j_3}} = 0, \\ i\partial_T A_- + i\partial_\ell \omega^-(0)\partial_X A_- + \sum_{j_1,j_2,j_3\in\{+,-\}} \nu^-_{j_1 j_2 j_3} A_{j_1} A_{j_2} \overline{A_{j_3}} = 0, \end{cases}$$

Theorem (Gilg–Schneider-P, 2016)

For every $C_0 > 0$ and $T_0 > 0$, there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all solutions $A_\pm \in C(\mathbb{R}, H^2(\mathbb{R}))$ of the Dirac equations with

$$\sup_{T\in[0,T_0]} \|A_\pm(T,\cdot)\|_{H^2} \leq C_0$$

and for all $\varepsilon \in (0,\varepsilon_0)$, there are solutions $U \in C([0,T_0/\varepsilon^2], L^\infty(\mathbb{R}))$ of the NLS equation on the periodic graph Γ satisfying the bound

$$\sup_{t\in[0,T_0/\varepsilon^2]} \sup_{x\in\mathbb{R}} |U(t,x) - \varepsilon\Psi_{\text{dirac}}(t,x)| \leq C\varepsilon^{3/2}.$$

Function spaces

The operator $L = -\partial_x^2$ is considered in the space

$$\mathcal{L}^2 = \{U = (u_0, u_+, u_-) \in (L^2(\mathbb{R}))^3 : \quad \mathrm{supp}(u_{n,j}) = I_{n,j}, \quad n \in \mathbb{Z}, \;\; j \in \{0, +, -\}\}$$

with the domain of definition

$$\mathcal{H}^2 \;\; := \;\; \{U \in \mathcal{L}^2 : \;\; u_{n,j} \in H^2(I_{n,j}), \quad n \in \mathbb{Z}, \;\; j \in \{0, +, -\} \;\; \text{Kirchhoff BCs}\}.$$

Function spaces

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- The space $\mathcal{H}^2$ is closed under pointwise multiplication.
- The skew symmetric operator $-iL$ defines a unitary semi-group $(e^{-iLt})_{t\in\mathbb{R}}$ in $\mathcal{L}^2$.
- There exists a positive constant C_L such that
$$\|e^{-iLt}U\|_{\mathcal{H}^2} \leq C_L\|U\|_{\mathcal{H}^2}$$
for every $U \in \mathcal{H}^2$ and every $t \in \mathbb{R}$.
- There exists a unique local solution $U \in C([-T_0, T_0], \mathcal{H}^2)$ to the NLS equation on the periodic graph Γ.

Bloch transform on the real line

For a function $f : \mathbb{R} \to \mathbb{C}$, Bloch transform is defined by

$$\widetilde{f}(\ell, x) = (\mathcal{T} f)(\ell, x) = \sum_{j \in \mathbb{Z}} e^{ijx} \widehat{f}(\ell + j),$$

where $\widehat{f}(\xi) = (\mathcal{F} f)(\xi)$, $\xi \in \mathbb{R}$ is the Fourier transform of f. The inverse transform is

$$f(x) = (\mathcal{T}^{-1} \widetilde{f})(x) = \int_{-1/2}^{1/2} e^{i\ell x} \widetilde{f}(\ell, x) d\ell.$$

By construction, $\widetilde{f}(\ell, x)$ is extended from $(\ell, x) \in \mathbb{T}_1 \times \mathbb{T}_{2\pi}$ to $(\ell, x) \in \mathbb{R} \times \mathbb{R}$ according to the continuation conditions:

$$\widetilde{f}(\ell, x) = \widetilde{f}(\ell, x + 2\pi) \quad \text{and} \quad \widetilde{f}(\ell, x) = \widetilde{f}(\ell + 1, x) e^{ix}.$$

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where $\widehat{f}(\xi) = (\mathcal{F} f)(\xi)$, $\xi \in \mathbb{R}$ is the Fourier transform of f. The inverse transform is

$$f(x) = (\mathcal{T}^{-1} \widetilde{f})(x) = \int_{-1/2}^{1/2} e^{i\ell x} \widetilde{f}(\ell, x) d\ell.$$

By construction, $\widetilde{f}(\ell, x)$ is extended from $(\ell, x) \in \mathbb{T}_1 \times \mathbb{T}_{2\pi}$ to $(\ell, x) \in \mathbb{R} \times \mathbb{R}$ according to the continuation conditions:

$$\widetilde{f}(\ell, x) = \widetilde{f}(\ell, x + 2\pi) \quad \text{and} \quad \widetilde{f}(\ell, x) = \widetilde{f}(\ell + 1, x) e^{ix}.$$

- $\mathcal{T}$ is an isomorphism between $H^s(\mathbb{R})$ and $L^2(\mathbb{T}_1, H^s(\mathbb{T}_{2\pi}))$.
- Multiplication in x space corresponds to convolution in Bloch space.
- If $\chi : \mathbb{R} \to \mathbb{R}$ is 2π periodic, then

  $$\mathcal{T}(\chi u)(\ell, x) = \chi(x)(\mathcal{T} u)(\ell, x).$$

  In particular, if χ_j are periodic cut-off functions in I_j, $j \in \{0, +, -\}$, then $\mathcal{T}(u_j)(\ell, x) = \mathcal{T}(\chi_j u_j)(\ell, x) = \chi_j(x)(\mathcal{T} u_j)(\ell, x)$.

Function spaces for Bloch transforms

The operator $\tilde{L}(\ell) = -(\partial_x + i\ell)^2$ is self-adjoint in the space

$$L^2_\Gamma := \{ \ \widetilde{U} = (\widetilde{u}_0, \widetilde{u}_+, \widetilde{u}_-) \in (L^2(\mathbb{T}_{2\pi}))^3 : \quad \mathrm{supp}(\widetilde{u}_j) = I_{0,j}, \quad j \in \{0,+,-\}\}$$

with the domain of definition

$$H^2_\Gamma := \{\widetilde{U} \in L^2_\Gamma : \ \ \widetilde{u}_j \in H^2(I_{0,j}), \quad j \in \{0,+,-\}, \qquad \text{Kirchhoff BCs}\}.$$

In Bloch space, we work with functions in $L^2(\mathbb{T}_1, L^2_\Gamma)$. Local well-posedness applies to smooth functions in $\widetilde{\mathcal{H}}^2 = L^2(\mathbb{T}_1, H^2_\Gamma)$.

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Key Lemma: *The Bloch transform $\mathcal{T}$ is an isomorphism between $\mathcal{H}^2$ and $\widetilde{\mathcal{H}}^2$.*

Rest of the proof

- Bloch transform for the NLS equation on the periodic graph Γ.
- Decomposition of solutions in the Bloch space

$$\widetilde{U}(t,\ell,x) = \widetilde{V}(t,\ell) f^{(m_0)}(\ell,x) + \widetilde{U}^{\perp}(t,\ell,x)$$

- Approximation of the principal part of the solution

$$\widetilde{V}_{\mathrm{app}}(t,\ell) = \widetilde{A}\left(\varepsilon^2 t, \frac{\ell-\ell_0}{\varepsilon}\right) e^{-i\omega^{(m_0)}(\ell_0)t} e^{-i\partial_\ell \omega^{(m_0)}(\ell_0)(\ell-\ell_0)t}.$$

  As $\varepsilon \to 0$, $\widetilde{A}$ satisfies the homogeneous NLS equation in the Fourier space.

- A near-identity transformation for $\widetilde{U}^{\perp}(t,\ell,x)$ with a suitable chosen approximation $\widetilde{U}^{\perp}_{\mathrm{app}}(t,\ell,x)$.
- Estimates of residual terms in Bloch spaces.
- Estimates of the approximation between the Fourier space and Bloch space.
- Estimates of the error term in time evolution with Gronwall's inequality.

Homogeneous NLS equation

The asymptotic solution in the form

$$U(t,x) = \varepsilon A(T,X) f^{(m_0)}(\ell_0,x) e^{i\ell_0 x} e^{-i\omega^{(m_0)}(\ell_0)t} + \text{higher-order terms},$$

with $T = \varepsilon^2 t$ and $X = \varepsilon(x - c_g t)$ satisfies the homogeneous NLS equation

$$i\partial_T A + \frac{1}{2}\partial_\ell^2 \omega^{(m_0)}(\ell_0)\partial_X^2 A + \nu |A|^2 A = 0, \quad \nu = \frac{\|f^{(m_0)}(\ell_0,\cdot)\|^4_{L^4_{\rm per}}}{\|f^{(m_0)}(\ell_0,\cdot)\|^2_{L^2_{\rm per}}}.$$

Theorem (Gilg–Schneider-P, 2016)

Fix $m_0 \in \mathbb{N}$, $\ell_0 \in \mathbb{T}_1$, and assume $\omega^{(m)}(\ell_0) \neq \omega^{(m_0)}(\ell_0)$ for every $m \neq m_0$. Then, for every $C_0 > 0$ and $T_0 > 0$, there exist $\varepsilon_0 > 0$ and $C > 0$ such that for all solutions $A \in C(\mathbb{R}, H^3(\mathbb{R}))$ of the homogeneous NLS equation with

$$\sup_{T\in[0,T_0]} \|A(T,\cdot)\|_{H^3} \leq C_0$$

and for all $\varepsilon \in (0,\varepsilon_0)$, there are solutions $U \in C([0,T_0/\varepsilon^2], L^\infty(\mathbb{R}))$ to the NLS equation on the periodic graph Γ satisfying the bound

$$\sup_{t\in[0,T_0/\varepsilon^2]} \sup_{x\in\mathbb{R}} \left| U(t,x) - \varepsilon A(T,X) f^{(m_0)}(\ell_0,x) e^{i\ell_0 x} e^{-i\omega^{(m_0)}(\ell_0)t} \right| \leq C\varepsilon^{3/2}.$$

Bifurcations of stationary states

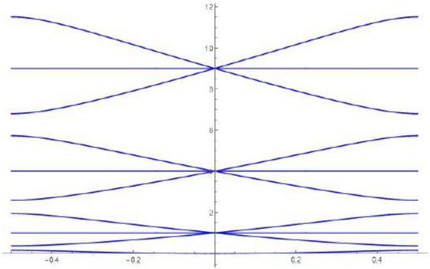
The stationary NLS equation on the periodic graph Γ:

$$-\partial_x^2\phi - 2|\phi|^2\phi = \Lambda\phi \qquad \Lambda \in \mathbb{R}, \quad \phi(x) : \Gamma \to \mathbb{R}.$$

The effective homogeneous NLS equation on the real line

$$-\frac{1}{2}\partial_\ell^2\omega^{(m_0)}(\ell_0)\partial_X^2 A - \nu|A|^2 A = \Omega A, \quad A(X) : \mathbb{R} \to \mathbb{R}.$$

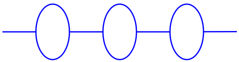
The stationary reduction is satisfied if $\partial_\ell\omega^{(m_0)}(\ell_0) = 0$.



Nonlinear bound states on the periodic graph

Stable bound states bifurcate from the bottom of the linear spectrum at $\Lambda = 0$:

$$-\partial_x^2 \phi - 2|\phi|^2 \phi = \Lambda \phi \qquad \Lambda \in \mathbb{R}, \quad \phi(x) : \Gamma \to \mathbb{R}.$$



Theorem

There are positive constants Λ_0 and C_0 such that for every $\Lambda \in (-\Lambda_0, 0)$, there exist two bound states $\phi \in \mathcal{D}(\Gamma)$ (up to the discrete translational invariance) s.t. either

$$\phi(x - L/2) = \phi(L/2 - x), \quad x \in \Gamma$$

or

$$\phi(x - L - \pi/2) = \phi(L + \pi/2 - x), \quad x \in \Gamma.$$

Moreover, it is true for both bound states that

(i) *ϕ is symmetric in upper and lower semicircles of Γ,*

(ii) *$\phi(x) > 0$ for every $x \in \Gamma$,*

(iii) *$\phi(x) \to 0$ as $|x| \to \infty$ exponentially fast.*

Numerical approximations of the bound states with $L = \pi$

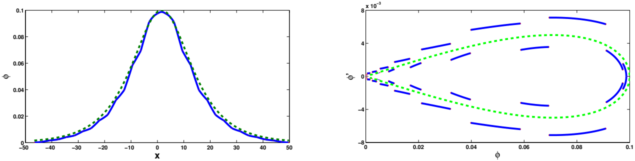


Figure: Profile of the numerically generated bound state on (x, ϕ) plane (left) and on (ϕ, ϕ') plane (right). The red dots show the break points on the periodic graph Γ. The green dashed line shows the NLS soliton on the infinite line.

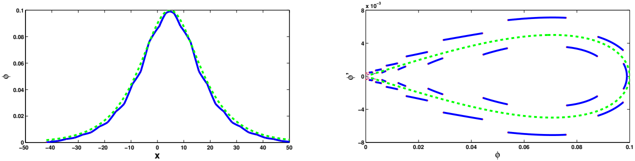


Figure: The same but for the other bound state.

Homogenization of the discrete map

We set $\Lambda = -\epsilon^2$ and consider the limit $\epsilon \to 0$.

For every $(a, b) \in \mathbb{R}^2$ and every $\epsilon \in \mathbb{R}$, there is a unique solution $\psi(x; a, b, \epsilon) \in C^\infty(\mathbb{R})$ of the initial-value problem:

$$\left\{ \begin{array}{ll} \partial_x^2 \psi - \epsilon^2 \psi + 2|\psi|^2 \psi = 0, & x \in \mathbb{R}, \\ \psi(0) = a, & \\ \partial_x \psi(0) = b, & \end{array} \right.$$

For each $\Gamma_{n,0}$ and $\Gamma_{n,\pm}$, the solution can be defined in the implicit form:

$$\phi_{n,0}(x) = \psi(x - 2\pi n; a_n, b_n, \epsilon), \quad \phi_{n,\pm}(x) = \psi(x - 2\pi n - \pi; c_n, d_n, \epsilon).$$

Kirchhoff boundary conditions produces a two-dimensional map:

$$\left\{ \begin{array}{l} a_{n+1} = \psi(\pi; c_n, d_n, \epsilon), \\ b_{n+1} = 2\partial_x \psi(\pi; c_n, d_n, \epsilon), \end{array} \right. \qquad \left\{ \begin{array}{l} c_n = \psi(\pi; a_n, b_n, \epsilon), \\ 2d_n = \partial_x \psi(\pi; a_n, b_n, \epsilon), \end{array} \right.$$

The nonlinear discrete map generalizes the linear transfer matrix method.

Approximate continuous solution

In the limit $\epsilon \to 0$, expand solution $\psi(x; \epsilon\alpha, \epsilon^2\beta, \epsilon)$ in the power series in ϵ.

$$\left\{ \begin{array}{l} \alpha_{n+1} = \alpha_n + \frac{3}{2}\epsilon\pi\beta_n + \frac{3}{2}\epsilon^2\pi^2(1 - 2\alpha_n^2)\alpha_n + \mathcal{O}(\epsilon^3), \\ \beta_{n+1} = \beta_n + \epsilon 3\pi(1 - 2\alpha_n^2)\alpha_n + \frac{7}{4}\epsilon^2\pi^2(1 - 6\alpha_n^2)\beta_n + \mathcal{O}(\epsilon^3). \end{array} \right.$$

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Approximate continuous solution:

$$\alpha_n = A(X+X_0), \quad \beta_n = B(X+X_0), \quad X = \epsilon n, \quad n \in \mathbb{Z},$$

where X_0 is arbitrary and A, B satisfy the continuous limit

$$\begin{cases} A'(X) = 3\pi/2B(X), \\ B'(X) = 3\pi(1-2A^2)A(X), \end{cases}$$

with the continuous NLS solitons

$$A(X) = \mathrm{sech}(\nu X), \quad B(X) = -\mu \tanh(\nu X)\mathrm{sech}(\nu X), \quad X \in \mathbb{R},$$

Justification of the approximate continuous solution

Key Lemma: *For a given $f \in \ell^2(\mathbb{Z})$ satisfying the reversibility symmetry $f_n = f_{1-n}$ for every $n \in \mathbb{Z}$, consider solutions of the linearized difference equation*

$$-\frac{\alpha_{n+1} - 2\alpha_n + \alpha_{n-1}}{\epsilon^2} + \nu^2(1 - 6A^2(\epsilon n))\alpha_n = f_n, \quad n \in \mathbb{Z}.$$

For sufficiently small $\epsilon > 0$, there exists a unique solution $\alpha \in \ell^2(\mathbb{Z})$ satisfying the reversibility symmetry $\alpha_n = \alpha_{1-n}$ for every $n \in \mathbb{Z}$. Moreover there is a positive ϵ-independent constant C such that

$$\epsilon^{-1} \|\sigma_+\alpha - \alpha\|_{\ell^2} \leq C\|f\|_{\ell^2}, \quad \|\alpha\|_{\ell^2} \leq C\|f\|_{\ell^2},$$

where σ_+ is the shift operator defined by $(\sigma_+\alpha)_n := \alpha_{n+1}$, $n \in \mathbb{Z}$.

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where σ_+ *is the shift operator defined by* $(\sigma_+\alpha)_n := \alpha_{n+1}$*,* $n \in \mathbb{Z}$*.*

- Translational parameter X_0 can be chosen to satisfy the reversibility symmetry.
- Two reversibility symmetries give two nonlinear bound states.
- The symmetry $\phi_+ = \phi_-$ holds by construction.
- Positivity and exponential decay are not obtained from this method.

Positivity and exponential decay

The perturbative two-dimensional map:

$$\begin{cases} \alpha_{n+1} = \alpha_n + \frac{3}{2}\epsilon\pi\beta_n + \frac{3}{2}\epsilon^2\pi^2(1-2\alpha_n^2)\alpha_n + \mathcal{O}(\epsilon^3), \\ \beta_{n+1} = \beta_n + \epsilon 3\pi(1-2\alpha_n^2)\alpha_n + \frac{7}{4}\epsilon^2\pi^2(1-6\alpha_n^2)\beta_n + \mathcal{O}(\epsilon^3). \end{cases}$$

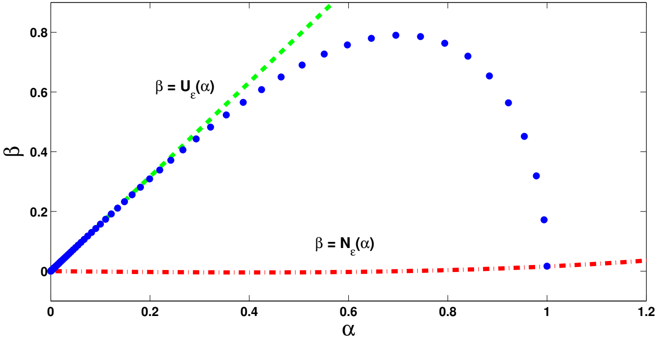
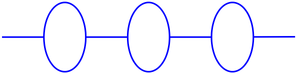


Figure: The plane (α, β), where the blue dots denote a sequence $\{\alpha_n, \beta_n\}_{n\in\mathbb{Z}}$, the green dashed line shows the unstable curve $\beta = \mathcal{U}_\epsilon(\alpha)$, and the red dash-dotted line shows the symmetry curve $\beta = \mathcal{N}_\epsilon(\alpha)$.

Conclusion



For the periodic graph Γ, we have obtained the following results:

- We developed the Bloch transform on Γ and justified homogenization of the NLS equation on Γ with the homogeneous NLS or Dirac equations on the line.
- We approximated stationary states near the lowest spectral band by using NLS solitons.
- Scattering dynamics and ground state properties are still opened on the periodic graph Γ.

Thank you!