Instability of peaked periodic waves in the reduced Ostrovsky equation

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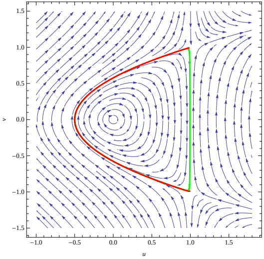

Joint work with Anna Geyer
(Delft University of Technology, Netherlands)

Introduction

The generalized *reduced Ostrovsky equation*

$$(u_t + u^p u_x)_x = u,$$

where u is a real-valued function of (x,t) and $p \in \mathbb{N}$.

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- For $p = 1$, the equation arises as $\beta \to 0$ from the Ostrovsky equation

  $$(u_t + uu_x + \beta u_{xxx})_x = \gamma u$$

  derived in the context of long gravity waves in a rotating fluid, as a generalization of the KdV equation ($\gamma = 0$). [Ostrovsky, 1978]

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- Local well-posedness in H^s for $s > 3/2$. [Stefanov et. al., 2010]

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- ▷ Zero mass constraint is necessary: $\int u dx = 0$.

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- ▷ The equations can be transformed to an integrable equation of Klein–Gordon type by a solution-dependent coordinate change.
  [Vakhnenko & Parkes, 1998], [Kraenkel & Leblond & Manna 2014]
- ▷ For $p = 1$: explicit periodic traveling waves exist;
  smooth solutions in terms of Jacobi elliptic functions
  [Grimshaw & Helfrich & Johnson 2012],
  peaked solutions with parabolic shape [Ostrovsky, 1978]

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- Any $p \in N$: Spectral stability of smooth periodic waves for *co-periodic* perturbations. [Geyer & P., 2017]

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Next goal: *Linear and nonlinear instability* of the limiting peaked periodic wave for $p = 1$.

Traveling wave solutions

Traveling wave solutions are solutions of the form

$$u(x,t) = U(x - ct),$$

where $z = x - ct$ is the travelling wave coordinate and $c > 0$ is the wave speed. The wave profile U is $2T$-periodic for fixed $c > 0$.

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The wave profile U satisfies the boundary-value problem

$$\frac{d}{dz}\left((c - U^p)\frac{dU}{dz}\right) + U(z) = 0, \qquad \left.\begin{array}{l} U(-T) = U(T), \\ U'(-T) = U'(T), \end{array}\right\} \quad \text{(ODE)}$$

where $\int_{-T}^{T} U(z)dz = 0$, i.e. the periodic waves have zero mean.

Existence of periodic traveling waves

Let $c > 0$ and $p \in \mathbb{N}$. A function U is a smooth periodic solution of

$$\frac{d}{dz}\left((c - U^p)\frac{dU}{dz}\right) + U = 0 \qquad \text{(ODE)}$$

iff $(u, v) = (U, U')$ is a periodic orbit γ_E of the planar system

$$\begin{cases} u' = v, \\ v' = \dfrac{-u + pu^{p-1}v^2}{c - u^p}, \end{cases}$$

which has the first integral

$$E(u, v) = \frac{1}{2}(c - u^p)^2 v^2 + \frac{c}{2}u^2 - \frac{1}{p+2}u^{p+2}.$$

The periodic wave U is smooth iff $c - U(z)^p > 0$ for every z.

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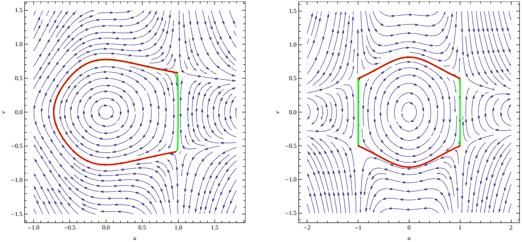

There exists a smooth family of periodic solutions $U \in \dot{H}^\infty_{\text{per}}$ of (ODE) parametrized by the energy $E \in (0, E_c)$, where $2T$ depends on E.

Scaling transformation

For fixed c, the map $E \mapsto T$ is decreasing with $T(0) = \pi c^{1/2}$.
For fixed T, the map $E \mapsto c$ is increasing with $c(0) = T^2/\pi^2$.

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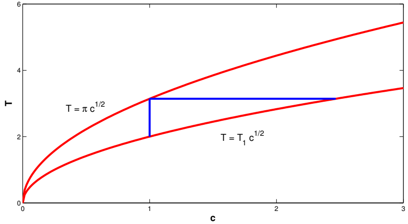
The map $E \mapsto T$ for fixed c is transferred to the map $E \mapsto c$ for fixed T by the scaling transformation

$$U(z; c) = c^{1/p} \tilde{U}(\tilde{z}), \quad z = c^{1/2} \tilde{z}, \quad T = c^{1/2} \tilde{T},$$

where $\tilde{U}$ is a $2\tilde{T}$-periodic solution of the same (ODE) with $c = 1$.



Peaked periodic wave for $p = 1$

The 2π periodic traveling wave solutions $U(z)$ satisfy the BVP

$$\left\{ \begin{array}{l} [c - U(z)]\, U'(z) + (\partial_z^{-1} U)(z) = 0, \quad z \in (-\pi, \pi) \\ U(-\pi) = U(\pi), \end{array} \right.$$

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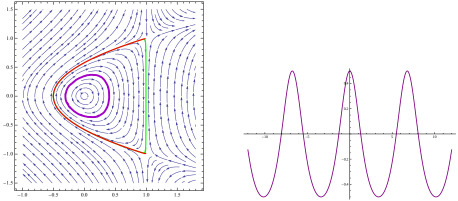
Lemma (Existence of smooth periodic waves)

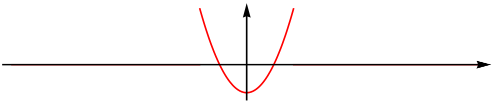
There exists $c_* > 1$ such that for every $c \in (1, c_*)$, the BVP admits a unique smooth periodic wave U satisfying $U(z) < c$ for $z \in [-\pi, \pi]$.



Peaked periodic wave for $p = 1$

For $c = c_* := \pi^2/9$ there exists a solution with parabolic profile

$$U_*(z) := \frac{3z^2 - \pi^2}{18}, \quad z \in [-\pi, \pi],$$



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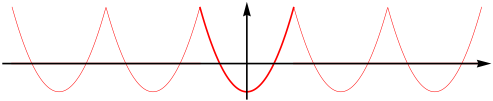

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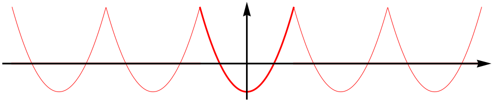

▷ The peaked periodic wave $U_* \in \dot{H}^s_{\rm per}(-\pi, \pi)$ for $s < 3/2$:

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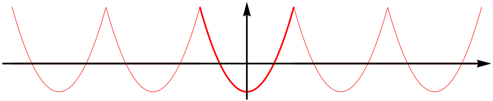

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▷ $U_*(z) < c_*$ for $z \in (-\pi, \pi)$, $U_*(\pm\pi) = c_*$, and $U_*'(\pm\pi) = \pm\pi/3$.

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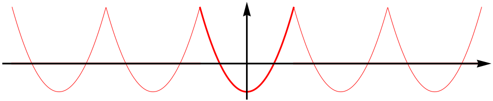

Lemma

The peaked periodic wave U_* is the unique *solution with a jump discontinuity in the derivative at $z = \pm\pi$.*

Broader picture on stability of peaked periodic waves

- KdV equation: smooth solutions are stable, no peaked solutions
  [Deconinck et. al. 2009,2010]

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- Ostrovsky equation: all smooth solutions are stable, but the limiting *peaked solution is unstable*.
  [Geyer & P. 2018]

Spectral stability of the peaked periodic wave

Let $u = U + v$ and consider the linearized evolution for a co-periodic perturbation v to the travelling wave U:

$$\left\{ \begin{array}{l} v_t + \partial_z \left[(U_*(z) - c_*) v \right] = \partial_z^{-1} v, \quad t > 0, \\ v|_{t=0} = v_0, \end{array} \right.$$

or equivalently

$$v_t = \partial_z L v, \quad \text{where} \;\; L = P_0 \left(\partial_z^{-2} + c_* - U_* \right) P_0 : \quad \dot{L}^2_{\rm per} \to \dot{L}^2_{\rm per},$$

where $\dot{L}^2_{\rm per}$ is the L^2 space of periodic function with zero mean.

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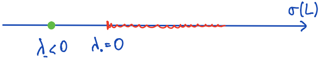
The spectrum of the self-adjoint operator L is $\sigma(L) = \{\lambda_-\} \cup \left[0, \frac{\pi^2}{6}\right]$.



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The spectral stability problem can not be solved by applying standard energy methods due to the lack of coercivity.

Linear stability of the peaked periodic wave

Consider the linearized evolution for a co-periodic perturbation v to the travelling wave U:

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Goal: show that the peaked periodic wave is *linearly unstable*.

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Definition

The travelling wave U is *linearly stable* if for every $v_0 \in \dot{H}^1_{\text{per}}$ there exists a unique global solution $v \in C(\mathbb{R}, \dot{H}^1_{\text{per}})$ to (linO) s.t.

$$\|v(t)\|_{H^1_{\text{per}}} \leq C \|v_0\|_{H^1_{\text{per}}}, \quad t > 0.$$

Otherwise, it is said to be linearly unstable.

Linear instability of the peaked periodic wave

▷ **Step 1**: The *truncated problem*

$$\begin{cases} v_t + \frac{1}{6}\partial_z \left[(z^2 - \pi^2)v\right] = 0, \quad t > 0, \\ v|_{t=0} = v_0 \in \dot{H}^1_{\rm per}. \end{cases} \qquad \text{(truncO)}$$

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Method of characteristics. The characteristic curves $z = Z(s,t)$ are found explicitly and the solution of $V(s,t) := v(Z(s,t),t)$ is

$$V(s,t) = \frac{1}{\pi^2}[\pi \cosh(\pi t/6) - s\sinh(\pi t/6)]^2 v_0(s), \quad s \in [-\pi,\pi], \quad t \in \mathbb{R}.$$

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This yields the linear instability result for the truncated problem:

Lemma

For every $v_0 \in \dot{H}^1_{\rm per}$ $\exists!$ global solution $v \in C(\mathbb{R}, \dot{H}^1_{\rm per})$ to (truncO).
If v_0 is odd, then the global solution satisfies

$$\frac{1}{2}\|v_0\|_{L^2} e^{\pi t/6} \le \|v(t)\|_{L^2} \le \|v_0\|_{L^2} e^{\pi t/6}, \quad t > 0.$$

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Lemma

For every $v_0 \in \dot{H}^1_{\rm per}$ $\exists!$ *global solution* $v \in C(\mathbb{R}, \dot{H}^1_{\rm per})$ *to* (linO).
If v_0 *is odd, then the solution satisfies*

$$C\|v_0\|_{L^2} e^{\pi t/6} \leq \|v(t)\|_{L^2} \leq \|v_0\|_{L^2} e^{\pi t/6}, \quad t>0.$$

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Conclusion: The reduced Ostrovsky equation is *linearly unstable*.

Nonlinear instability

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- ▷ True in finite dimensional case
- ▷ In infinite dimensions:

$$v_t = Av + F(v)$$

A is a linear operator generating a C^0-semigroup in Banach space X and F is strongly continuous in X

If A has positive spectrum $\{\mathcal{R}\lambda > 0\}$,
then $v = 0$ is nonlinearly unstable. [Shatah & Strauss '00]

Nonlinear instability

Does *linear* instability of the peaked periodic wave U_* *imply nonlinear* instability?

- True in finite dimensional case
- In infinite dimensions:

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  A is a linear operator generating a C^0-semigroup in Banach space X and F is strongly continuous in X

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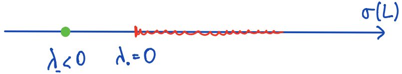
- Here: $A = \partial_z L$ but



  so we do not know whether the spectral assumption is satisfied.

- We need a different approach!

Nonlinear instability

Consider an orbit $\{U_*(z-a), \;\; a \in [-\pi, \pi]\}$ of the peaked wave U_*.

Nonlinear instability

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Definition

The travelling wave U is said to be *orbitally stable* if for every $\epsilon > 0$, there exists $\delta > 0$ such that

for every $u_0 \in \dot{H}^1_{\rm per}$ satisfying $\|u_0 - U\|_{H^1_{\rm per}} < \delta$,

there exists a unique global solution $u \in C(\mathbb{R}, \dot{H}^1_{\rm per})$ to

$$\begin{cases} u_t + uu_x = \partial_x^{-1} u, & t > 0, \\ u|_{t=0} = u_0, \end{cases} \qquad \text{(redO)}$$

such that for every $t > 0$,

$$\inf_{a \in [-\pi,\pi]} \|u(t,\cdot) - U(\cdot - a)\|_{H^1_{\rm per}} < \epsilon.$$

Otherwise, the periodic wave U is said to be orbitally unstable.

Nonlinear instability

We consider *decomposition of the solution* $u \in \dot{H}^1_{\rm per}$

$$u(t,x) = U_*(x - ct - a(t)) + v(t, x - ct - a(t)),$$

for a co-periodic perturbation v satisfying the *orthogonality condition*

$$\langle \partial_x U_*, v \rangle_{L^2} = 0.$$

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Such a decomposition always exists and is unique by an application of the inverse function theorem.

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for a co-periodic perturbation v satisfying (CPv):

$$\begin{cases} v_t + \frac{1}{6}\partial_z \left[(z^2 - \pi^2) v \right] + v \partial_z v = \partial_z^{-1} v + a'(t)(\partial_z U_* + \partial_z v), \\ v|_{t=0} = v_0, \end{cases}$$

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Using the *orthogonality condition* we obtain an evolution equation

for the translation parameter a:

$$\begin{cases} a'(t) = -\frac{\langle \partial_z U, \partial_z L v \rangle_{L^2} - \langle \partial_z U, v \partial_z v \rangle_{L^2}}{\|\partial_z U\|^2_{L^2} + \langle \partial_z U, \partial_z v \rangle_{L^2}}, \quad t > 0, \\ a(0) = 0. \end{cases} \qquad \text{(CPa)}$$

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For local existence, we need $v \in \dot{H}^s_{\text{per}}$ with $s > 3/2$.

Nonlinear instability

Theorem (Orbital instability)

There exists $\epsilon > 0$ *such that for every small* $\delta > 0$,

there exists $v_0 \in \dot{H}^s_{\text{per}}$ *satisfying*

$$\|v_0\|_{H^s_{\text{per}}} \leq \delta$$

s.t. the unique solution $v \in C([0,T], \dot{H}^s_{\text{per}})$ to (CPv)–(CPa) satisfies

$$\|v(t_1)\|_{L^2} \geq \epsilon$$

for some $t_1 \in (0,T)$ *with* $T = \mathcal{O}(\delta^{-1})$, $a \in C([0,T],\mathbb{R})$ *and* $s > 3/2$.

Nonlinear instability – Proof

- Write (CPv)

$$\begin{cases} v_t + \frac{1}{6}\partial_z\left[(z^2-\pi^2)v\right] + v\partial_z v = \partial_z^{-1} v + a'(t)(\partial_z U_* + \partial_z v), \\ v|_{t=0} = v_0, \end{cases}$$

as the inhomogeneous evolution equation

$$v_t = Av + F(v)$$

where $A := A_0 + \partial_z^{-1}$ generates the C^0-semigroup in $\dot{L}^2_{\text{per}}$ and $F(v) : \dot{L}^2_{\text{per}} \to \dot{L}^2_{\text{per}}$ is continuous.

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- Every solution v to (CPv) satisfies the integral formulation

$$v(t) = S(t)v_0 + \int_0^t S(t-s)F(s)ds, \quad t \in [0, T].$$

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$$\|v(t)\|_{L^2} \geq C\|v_0\|_{L^2} e^{\pi t/6} - \int_0^t e^{\pi(t-t')/6} \left\|F(t')\right\|_{L^2} dt'$$

- Using the translation equation (CPa) for $a(t)$, we obtain that for any fixed $\varepsilon > 0$ there exists $t_1 \in [0,T]$ such that

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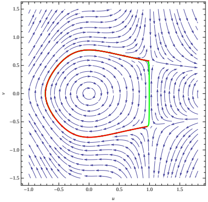
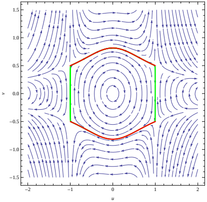
- This yields orbital instability of U_*.

Summary

- Periodic traveling waves of the reduced Ostrovsky equation

$$(u_t + u^p u_x)_x = u.$$





- The *smooth* periodic waves are spectrally *stable* for any $p \in \mathbb{N}$. [Geyer & P., LMP 2017]
- The *peaked* periodic wave is linearly and nonlinearly *unstable* for $p = 1$. [Geyer & P., SIMA 2018]