Reversing interfaces in systems with slow diffusion and strong absorption

Dmitry Pelinovsky
Department of Mathematics, McMaster University, Canada
http://dmpeli.math.mcmaster.ca

With Jamie Foster, University of Portsmouth, England

Macquarie University, 6 April 2018

The Linear Diffusion Equation with Linear Absorption

$$\frac{\partial h}{\partial t} = \frac{\partial^2 h}{\partial x^2} - h$$

The Slow Diffusion Equation with Strong Absorption

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(h^m \frac{\partial h}{\partial x}\right) - h^n$$

- Slow diffusion: $m > 0$
  implies finite propagation speed for contact lines
  (Herrero-Vazquez, 1987)

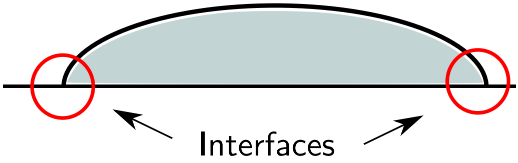
- Strong absorption: $n < 1$
  implies finite time extinction for compactly supported data
  (Kersner, 1983).

Physical Examples

The slow diffusion equation

$$\frac{\partial h}{\partial t} = \frac{\partial}{\partial x}\left(h^{m}\frac{\partial h}{\partial x}\right) - h^{n}$$

describes physical processes related to dynamics of interfaces.

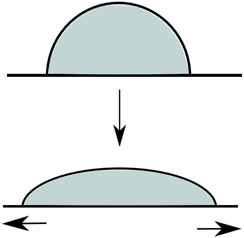


- spread of viscous films over a horizontal plate subject to gravity and constant evaporation ($m = 3$ and $n = 0$) (Acton-Huppert-Worster, 2001)
- dispersion of biological populations with a constant death rate ($m = 2$, $n = 0$)
- nonlinear heat conduction with a constant rate of heat loss ($m = 4$, $n = 0$)
- fluid flows in porous media with a drainage rate driven by gravity or background flows ($m = 1$ and $n = 1$ or $n = 0$) (Pritchard–Woods–Hogg, 2001)

Interface Dynamics

Advancing interfaces

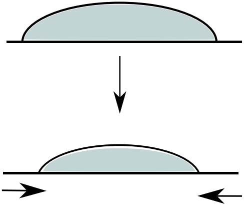
- driven by diffusion



$$h \sim (x - \ell(t))^{1/m}$$

Receding interfaces

- driven by absorption



$$h \sim (x - \ell(t))^{1/(1-n)}$$

Main objective: to construct self-similar solutions that exhibit reversing behaviour:

$$\text{Advancing} \to \text{Receding}$$

or anti-reversing behaviour:

$$\text{Receding} \to \text{Advancing}$$

Self-similar solutions

Consider the following self-similar reduction (Gandarias, 1994):

$$h(x,t) = (\pm t)^{\frac{1}{1-n}} H_\pm(\phi), \quad \phi = x(\pm t)^{-\frac{m+1-n}{2(1-n)}}, \quad \pm t > 0,$$

where $m > 0$ and $n < 1$. The functions $H_\pm$ satisfy the ODEs:

$$\frac{d}{d\phi}\left(H_\pm^m \frac{dH_\pm}{d\phi}\right) \pm \frac{m+1-n}{2(1-n)} \phi \frac{dH_\pm}{d\phi} = H_\pm^n \pm \frac{1}{1-n} H_\pm$$

Self-similar solutions

Consider the following self-similar reduction (Gandarias, 1994):

$$h(x,t) = (\pm t)^{\frac{1}{1-n}} H_\pm(\phi), \quad \phi = x(\pm t)^{-\frac{m+1-n}{2(1-n)}}, \quad \pm t > 0,$$

where $m > 0$ and $n < 1$. The functions $H_\pm$ satisfy the ODEs:

$$\frac{d}{d\phi}\left(H_\pm^m \frac{dH_\pm}{d\phi}\right) \pm \frac{m+1-n}{2(1-n)} \phi \frac{dH_\pm}{d\phi} = H_\pm^n \pm \frac{1}{1-n} H_\pm$$

We seek positive solutions $H_\pm$ on the semi-infinite line $[A_\pm, \infty)$ that satisfy

(i): $H_\pm(\phi) \to 0$ as $\phi \to A_\pm$,

(ii): $H_\pm(\phi)$ is monotonically increasing for all $\phi > A_\pm$,

(iii): $H_\pm(\phi) \to +\infty$ as $\phi \to +\infty$,

(iv): $H_+(\phi) \sim H_-(\phi)$ as $\phi \to +\infty$.

Self-similar solutions

Consider the following self-similar reduction (Gandarias, 1994):

$$h(x,t) = (\pm t)^{\frac{1}{1-n}} H_\pm(\phi), \quad \phi = x(\pm t)^{-\frac{m+1-n}{2(1-n)}}, \quad \pm t > 0,$$

where $m > 0$ and $n < 1$. The functions $H_\pm$ satisfy the ODEs:

$$\frac{d}{d\phi}\left(H_\pm^m \frac{dH_\pm}{d\phi}\right) \pm \frac{m+1-n}{2(1-n)}\phi\frac{dH_\pm}{d\phi} = H_\pm^n \pm \frac{1}{1-n}H_\pm$$

We seek positive solutions $H_\pm$ on the semi-infinite line $[A_\pm, \infty)$ that satisfy

(i): $H_\pm(\phi) \to 0$ as $\phi \to A_\pm$,

(ii): $H_\pm(\phi)$ is monotonically increasing for all $\phi > A_\pm$,

(iii): $H_\pm(\phi) \to +\infty$ as $\phi \to +\infty$,

(iv): $H_+(\phi) \sim H_-(\phi)$ as $\phi \to +\infty$.

If $A_\pm > 0$, the self-similar solutions exhibit the reversing behaviour:

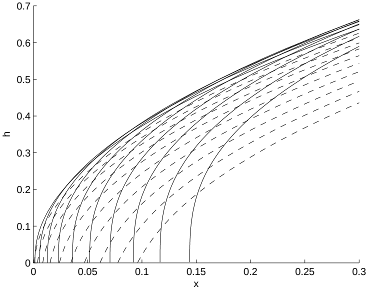
$$\ell(t) = A_\pm(\pm t)^{\frac{m+1-n}{2(1-n)}}, \quad \pm t > 0.$$

If $m + n > 1$, then $\ell'(0) = 0$.

Cartoon for reversing dynamics of an interface

Self-similar solution for reversing interface:

$$h(x,t) = (\pm t)^{\frac{1}{1-n}} H_\pm(\phi), \quad \phi = x(\pm t)^{-\frac{m+1-n}{2(1-n)}}, \quad \pm t > 0,$$

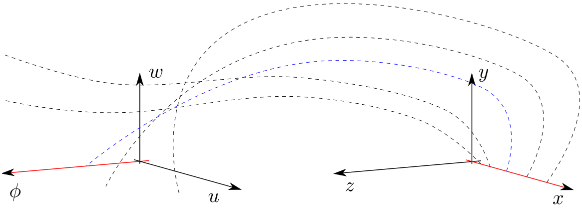


Dynamical Systems Framework

Solutions were approximated by a naive numerical scheme
in Foster *et al.* [SIAM J. Appl. Math. **72**, 144 (2012)].

The scope of our work is to develop a "rigorous" shooting method:

- The ODEs are singular in the limits of small and large $H_\pm$
- Make transformations to change singular boundary values to equilibrium points
- Obtain near-field asymptotics (small $H_\pm$): $(\phi, u, w) = (A_\pm, 0, 0)$
- Obtain far-field asymptotics (large $H_\pm$): $(x, y, z) = (x_0, 0, 0)$
- Connect between near-field and far-field asymptotics.



Near-field asymptotics

In variables $u = H_\pm$ and $w = H_\pm^m \frac{dH_\pm}{d\phi}$, the system is non-autonomous:

$$
\begin{aligned}
\frac{du}{d\phi} &= \frac{w}{u^m}, \\
\frac{dw}{d\phi} &= u^n \pm \frac{1}{1-n} u \mp \frac{m+1-n}{2(1-n)} \frac{\phi w}{u^m}.
\end{aligned}
$$

The system is singular at $u = 0$.

Near-field asymptotics

In variables $u = H_\pm$ and $w = H_\pm^m \frac{dH_\pm}{d\phi}$, the system is non-autonomous:

$$\begin{aligned}\frac{du}{d\phi} &= \frac{w}{u^m},\\ \frac{dw}{d\phi} &= u^n \pm \frac{1}{1-n}u \mp \frac{m+1-n}{2(1-n)}\frac{\phi w}{u^m}.\end{aligned}$$

The system is singular at $u = 0$.

Introduce the map $\tau \mapsto \phi$ by $\frac{d\phi}{d\tau} = u^m$ for $u > 0$. Then, we obtain the 3D autonomous dynamical system

$$\begin{cases}\dot{\phi} = u^m,\\ \dot{u} = w,\\ \dot{w} = u^{m+n} \pm \frac{1}{1-n}u^{m+1} \mp \frac{m+1-n}{2(1-n)}\phi w.\end{cases}$$

Near-field asymptotics

In variables $u = H_\pm$ and $w = H_\pm^m \frac{dH_\pm}{d\phi}$, the system is non-autonomous:

$$\begin{aligned} \frac{du}{d\phi} &= \frac{w}{u^m}, \\ \frac{dw}{d\phi} &= u^n \pm \frac{1}{1-n} u \mp \frac{m+1-n}{2(1-n)} \frac{\phi w}{u^m}. \end{aligned}$$

The system is singular at $u = 0$.

Introduce the map $\tau \mapsto \phi$ by $\frac{d\phi}{d\tau} = u^m$ for $u > 0$. Then, we obtain the 3D autonomous dynamical system

$$\begin{cases} \dot{\phi} = u^m, \\ \dot{u} = w, \\ \dot{w} = u^{m+n} \pm \frac{1}{1-n} u^{m+1} \mp \frac{m+1-n}{2(1-n)} \phi w. \end{cases}$$

The set of equilibrium points is given by $(\phi, u, w) = (A, 0, 0)$, where $A \in \mathbb{R}$. If $m > 1$, each equilibrium point is associated with the Jacobian matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \mp \frac{m+1-n}{2(1-n)} A \end{bmatrix}.$$

with a double zero eigenvalue and a simple nonzero eigenvalue if $A \neq 0$.

Center manifold

Case: $\pm A > 0$: the nonzero eigenvalue is negative. Trajectory cannot escape $(A, 0, 0)$ as $\tau \to -\infty$ along the stable curve. However, a two-dimensional center manifold exists:

$$W_c(A, 0, 0) = \left\{ w = \pm \frac{2(1-n)}{(m+1-n)A} u^{m+n} + \cdots, \quad \phi \in (A, A+\delta), \quad u \in (0, \delta) \right\}.$$

Dynamics on $W_c(A, 0, 0)$ is topologically equivalent to that of

$$\begin{cases} \dot{\phi} = u^m, \\ \dot{u} = \pm \frac{2(1-n)u^{m+n}}{(m+1-n)A}. \end{cases}$$

There exists exactly one trajectory on $W_c(A, 0, 0)$ which escapes the equilibrium point $(A, 0, 0)$ as $\tau \to -\infty$.

Center manifold

Case: $\pm A > 0$: the nonzero eigenvalue is negative. Trajectory cannot escape $(A, 0, 0)$ as $\tau \to -\infty$ along the stable curve. However, a two-dimensional center manifold exists:

$$W_c(A, 0, 0) = \left\{ w = \pm \frac{2(1-n)}{(m+1-n)A} u^{m+n} + \cdots, \quad \phi \in (A, A+\delta), \quad u \in (0, \delta) \right\}.$$

Dynamics on $W_c(A, 0, 0)$ is topologically equivalent to that of

$$\begin{cases} \dot{\phi} = u^m, \\ \dot{u} = \pm \frac{2(1-n)u^{m+n}}{(m+1-n)A}. \end{cases}$$

There exists exactly one trajectory on $W_c(A, 0, 0)$ which escapes the equilibrium point $(A, 0, 0)$ as $\tau \to -\infty$.

If $\pm A_\pm > 0$*, the unique solution has the following asymptotic behaviour*

$$H_\pm(\phi) = \left[\pm \frac{2(1-n)^2}{(m+1-n)A_\pm} (\phi - A_\pm) \right]^{1/(1-n)} + \cdots \quad \text{as} \quad \phi \to A_\pm.$$

The interface is driven by absorption.

Unstable manifold

Case: $\mp A > 0$: the center manifold is attracting (no trajectories escape $(A, 0, 0)$ as $\tau \to -\infty$). However, the nonzero eigenvalue is positive and a one-dimensional unstable curve exists:

$$W_u(A, 0, 0) = \left\{ \phi = A + \mathcal{O}(u^m), \quad w = \mp \frac{m + 1 - n}{2(1 - n)} Au + \mathcal{O}(u^{m+n}), \quad u \in (0, \delta) \right\}.$$

Dynamics on $W_u(A, 0, 0)$ is topologically equivalent to that of

$$\dot{u} = \mp \frac{m + 1 - n}{2(1 - n)} Au.$$

Unstable manifold

Case: $\mp A > 0$: the center manifold is attracting (no trajectories escape $(A, 0, 0)$ as $\tau \to -\infty$). However, the nonzero eigenvalue is positive and a one-dimensional unstable curve exists:

$$W_u(A, 0, 0) = \left\{ \phi = A + \mathcal{O}(u^m), \quad w = \mp \frac{m+1-n}{2(1-n)} Au + \mathcal{O}(u^{m+n}), \quad u \in (0, \delta) \right\}.$$

Dynamics on $W_u(A, 0, 0)$ is topologically equivalent to that of

$$\dot{u} = \mp \frac{m+1-n}{2(1-n)} Au.$$

If $\mp A_\pm > 0$, *the unique solution has the following asymptotic behaviour*

$$H_\pm(\phi) = \left(\mp \frac{m(m+1-n)A_\pm}{2(1-n)} (\phi - A_\pm) \right)^{1/m} + \cdots \quad \text{as} \quad \phi \to A_\pm.$$

The interface is driven by diffusion.

Far-field asymptotics

Question: If a trajectory departs from the point $(\phi, u, w) = (A, 0, 0)$, does it arrive to infinity: $\phi \to \infty$, $u \to \infty$?

Far-field asymptotics

Question: If a trajectory departs from the point $(\phi, u, w) = (A, 0, 0)$, does it arrive to infinity: $\phi \to \infty$, $u \to \infty$?

Let us change the variables

$$\phi = \frac{x}{y^{\frac{m+1-n}{2(1-n)}}}, \quad u = \frac{1}{y^{\frac{1}{1-n}}}, \quad w = \frac{z}{y^{\frac{m+3-n}{2(1-n)}}}$$

and re-parameterize the time τ with new time s by

$$\frac{d\tau}{ds} = y^{\frac{m+1-n}{2(1-n)}}, \quad y \geq 0.$$

The 3D autonomous dynamical system is rewritten as a smooth system

$$\begin{cases} x' = y - \frac{m+1-n}{2} xz, \\ y' = -(1-n)zy, \\ z' = \pm \frac{1}{1-n} y + y^2 \mp \frac{m+1-n}{2(1-n)} xz - \frac{m+3-n}{2} z^2, \end{cases}$$

The 3D smooth dynamical system is

$$\left\{ \begin{array}{l} x' = y - \frac{m+1-n}{2} xz, \\ y' = -(1-n)zy, \\ z' = \pm\frac{1}{1-n} y + y^2 \mp \frac{m+1-n}{2(1-n)} xz - \frac{m+3-n}{2} z^2, \end{array} \right.$$

The set of equilibrium points is given by $(x, y, z) = (x_0, 0, 0)$, where $x_0 \in \mathbb{R}$. Each equilibrium point is associated with the Jacobian matrix

$$\left[\begin{array}{ccc} 0 & 1 & -\frac{m+1-n}{2} x_0 \\ 0 & 0 & 0 \\ 0 & \pm\frac{1}{1-n} & \mp\frac{m+1-n}{2(1-n)} x_0 \end{array} \right].$$

with a double zero eigenvalue and a simple nonzero eigenvalue if $x_0 \neq 0$. Only $x_0 > 0$ is relevant for the asymptotics as $\phi \to +\infty$.

- Two-dimensional center manifold associated with the double zero eigenvalue.
- A stable curve for the upper sign and an unstable curve for the lower sign.

Center manifold

For every $x_0 > 0$, there exists a two-dimensional center manifold near $(x_0, 0, 0)$:

$$W_c(x_0, 0, 0) = \left\{ y = \frac{m+1-n}{2} xz + \mathcal{O}(z^2), \quad x \in (x_0 - \delta, x_0 + \delta), \;\; z \in (-\delta, \delta) \right\}.$$

The dynamics on $W_c(x_0, 0, 0)$ is topologically equivalent to that of

$$\left\{ \begin{array}{l} x' = \pm(1-n) \left(\frac{m+n+1}{2} - \frac{(m+1-n)^2}{4} x_0^2 \right) z^2, \\ z' = -(1-n) z^2. \end{array} \right.$$

There exists exactly one trajectory on $W_c(x_0, 0, 0)$, which approaches the equilibrium point $(x_0, 0, 0)$ as $s \to +\infty$.

Center manifold

For every $x_0 > 0$, there exists a two-dimensional center manifold near $(x_0, 0, 0)$:

$$W_c(x_0, 0, 0) = \left\{ y = \frac{m+1-n}{2} xz + \mathcal{O}(z^2), \quad x \in (x_0 - \delta, x_0 + \delta), \ \ z \in (-\delta, \delta) \right\}.$$

The dynamics on $W_c(x_0, 0, 0)$ is topologically equivalent to that of

$$\left\{ \begin{array}{l} x' = \pm(1-n) \left(\frac{m+n+1}{2} - \frac{(m+1-n)^2}{4} x_0^2 \right) z^2, \\ z' = -(1-n) z^2. \end{array} \right.$$

There exists exactly one trajectory on $W_c(x_0, 0, 0)$, which approaches the equilibrium point $(x_0, 0, 0)$ as $s \to +\infty$.

The solution at infinity satisfies the asymptotic behaviour

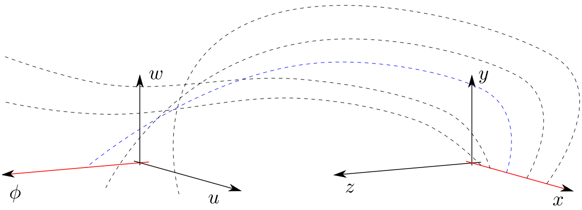
$$H_\pm(\phi) \sim \left(\frac{\phi}{x_0} \right)^{\frac{2}{m+1-n}} \quad \text{as} \quad \phi \to +\infty.$$

The family of diverging solutions is 1D for H_- and 2D for H_+.

Back to the plan

We are developing "rigorous" shooting method:

- The ODEs are singular in the limits of small and large $H_{\pm}$
- Make transformations to change singular boundary values to equilibrium points
- Obtain near-field asymptotics (small $H_{\pm}$): $(\phi, u, w) = (A_{\pm}, 0, 0)$
- Obtain far-field asymptotics (large $H_{\pm}$): $(x, y, z) = (x_0, 0, 0)$
- Connect between near-field and far-field asymptotics.



Connection results for H_+ (after reversing)

- Trajectory that departs from $(\phi, u, w) = (A_+, 0, 0)$ is 1D
- Trajectory that arrives to $(x, y, z) = (x_0, 0, 0)$ is 2D.

Fix $A_+ \in \mathbb{R}\backslash\{0\}$ *and consider a 1D trajectory such that* $(\phi, u, w) \to (A_+, 0, 0)$ *as* $\tau \to -\infty$ *and* $u > 0$. *Then, there exists a* $\tau_0 \in \mathbb{R}$ *such that* $\phi(\tau) \to +\infty$ *and* $u(\tau) \to +\infty$ *as* $\tau \to \tau_0$.

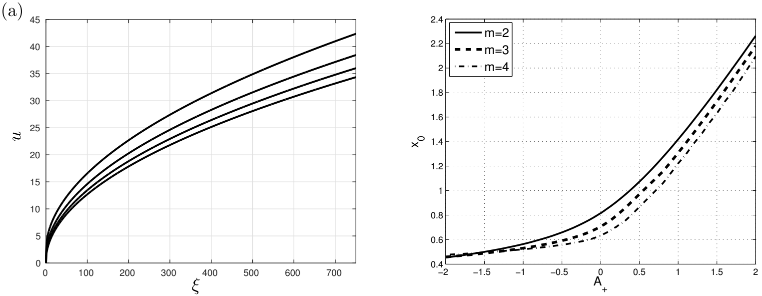


Figure: Left: trajectories with $m = 3$ and $n = 0$ for H_+. Right: variation of x_0 with A_+ for $m = 2,\ 3$ and 4.

Connection results for H_- (before reversing)

- Trajectory that departs from $(\phi, u, w) = (A_-, 0, 0)$ is 1D
- Trajectory that arrives to $(x, y, z) = (x_0, 0, 0)$ is 1D.

If we shoot from $(A_-, 0, 0)$, then the trajectory does not generally reach $(x_0, 0, 0)$.

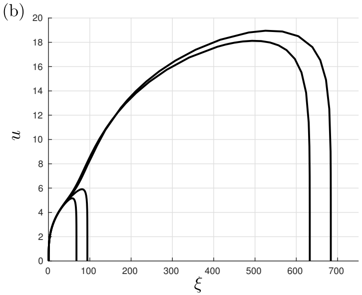


Figure: Trajectories with $m = 3$ and $n = 0$ for H_-.

Connection results for H_- (before reversing)

- Trajectory that departs from $(\phi, u, w) = (A_-, 0, 0)$ is 1D
- Trajectory that arrives to $(x, y, z) = (x_0, 0, 0)$ is 1D.

We should shoot from $(x_0, 0, 0)$ trying to reach $(A_-, 0, 0)$.

Fix $x_0 > 0$ *and consider a 1D trajectory such that* $(x, y, z) \to (x_0, 0, 0)$ *as* $s \to +\infty$ *and* $y > 0$. *There exists an* $s_0 \in \mathbb{R}$ *such that*

(i) *either* $w = 0$ *and* $u \geq 0$ *as* $s \to s_0$

(ii) *or* $u = 0$ *and* $w \geq 0$ *as* $s \to s_0$.

In both cases, if $(u, w) \neq (0, 0)$ *as* $s \to s_0$, *then* $|\phi| < \infty$ *as* $s \to s_0$.

Connection results for H_- (before reversing)

- Trajectory that departs from $(\phi, u, w) = (A_-, 0, 0)$ is 1D
- Trajectory that arrives to $(x, y, z) = (x_0, 0, 0)$ is 1D.

We should shoot from $(x_0, 0, 0)$ trying to reach $(A_-, 0, 0)$.

Fix $x_0 > 0$ and consider a 1D trajectory such that $(x, y, z) \to (x_0, 0, 0)$ as $s \to +\infty$ and $y > 0$. There exists an $s_0 \in \mathbb{R}$ such that

(i) *either $w = 0$ and $u \geq 0$ as $s \to s_0$*

(ii) *or $u = 0$ and $w \geq 0$ as $s \to s_0$.*

In both cases, if $(u, w) \neq (0, 0)$ as $s \to s_0$, then $|\phi| < \infty$ as $s \to s_0$.

Open ends:

- Do the two piecewise C^1 maps intersect?

$$\text{(i)} \quad \mathbb{R}^+ \ni x_0 \mapsto (\phi, u) \in \mathbb{R} \times \mathbb{R}^+ \quad \text{and} \quad \text{(ii)} \quad \mathbb{R}^+ \ni x_0 \mapsto (\phi, w) \in \mathbb{R} \times \mathbb{R}^+.$$

- If they do, does ϕ remain bounded at the intersection point?

And here the numerical approximation kicks in...

Finding the intersection points $x_0 = x_*$

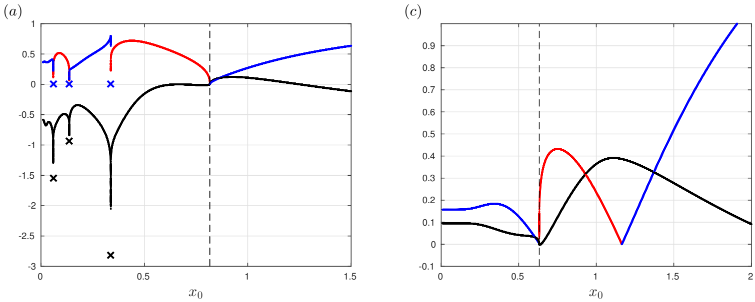
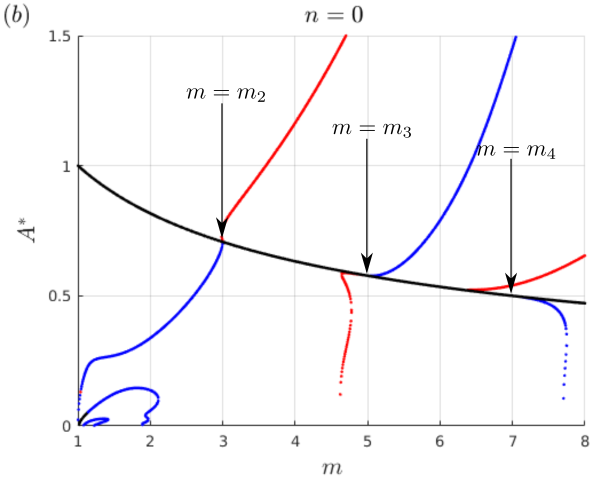


Figure: Panels (a)-(b) show plots of the piecewise C^1 maps for $m = 2$ and $m = 4$. In all cases the blue, red and black curves show the value of w at $u = 0$, the value of u at $w = 0$ and the value of ϕ at the termination point respectively.

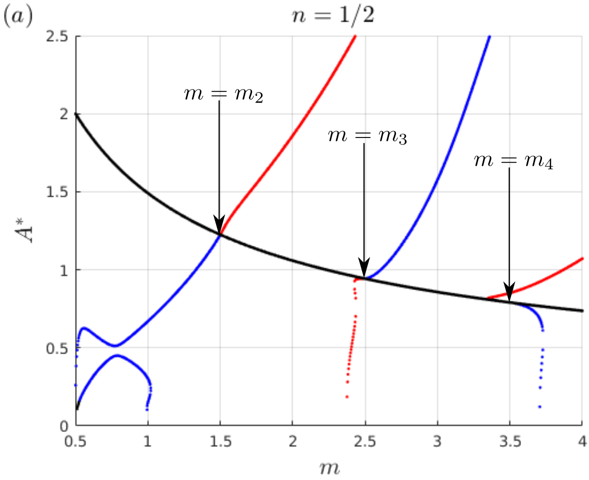
The dashed line corresponds to the exact solution with $A_- = 0$:

$$H_-(\phi) = \left(\frac{\phi}{x_*}\right)^{\frac{2}{m+1-n}}, \quad x_*^2 = \frac{2(m+1+n)}{(m+1-n)^2}.$$

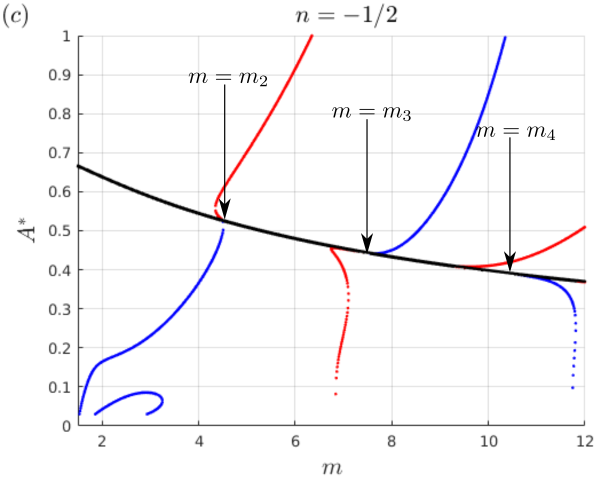
Self-similar solutions for $n = 0$



Self-similar solutions for other values of n



Self-similar solutions for other values of n



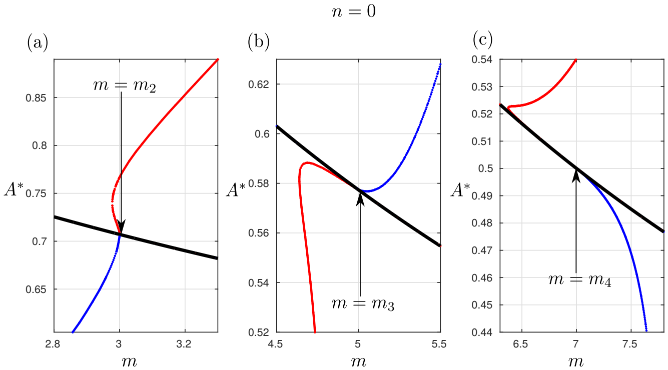
Location of bifurcations

The black curve corresponds to the exact solution with $A_- = 0$:

$$H_-(\phi) = \left(\frac{\phi}{x_*}\right)^{\frac{2}{m+1-n}}, \quad x_*^2 = \frac{2(m+1+n)}{(m+1-n)^2}.$$

After substituting self-similar variables, it is a static solution $h(x,t) = h(x)$. New self-similar solutions bifurcate from the static solutions at

$$m = m_k = (1-n)(2k-1), \quad k = 1, 2, 3, \dots$$



Analysis of bifurcations ($n = 0$)

Write H_- as a perturbation to the exact solution

$$H_- = r^{\frac{2}{m+1}} + u(r), \quad r := \frac{\phi}{x_*},$$

then u satisfies the homogeneous equation

$$\frac{m+1}{2}\frac{d^2u}{dr^2}\left(r^{\frac{2m}{m+1}}u(r)\right) - \frac{m+1}{2}r\frac{du}{dr} + u(r) = 0, \quad r \in (0,\infty).$$

Behavior at infinity:

$$u^{(I)}(r) \sim r^{\frac{2}{m+1}}, \quad u^{(II)}(r) \sim r^{-3}e^{\frac{m+1}{2}r^{2/(m+1)}} \quad \text{as} \quad r \to \infty$$

The admissible self-similar solutions must be proportional to $u^{(I)}$.

Near $r = 0$, the self-similar solutions satisfy

$$u(r) \sim c_1 r^{\frac{1-m}{1+m}} + c_2 r^{\frac{-2m}{1+m}} \quad \text{as} \quad r \to 0.$$

Both solutions diverge if $m > 1$ but the second solution diverges faster.

Kummer's differential equation

After a coordinate transformation, the homogeneous equation on $u(r)$ becomes the Kummer's differential equation (1837),

$$z\frac{d^2w}{dz^2} + (b-z)\frac{dw}{dz} + aw(z) = 0, \quad z \in (0, \infty),$$

where

$$a := -\frac{m+1}{2}, \quad b := \frac{m+3}{2}.$$

The power series solution is given by Kummer's function

$$M(z; a, b) = 1 + \frac{a}{b}\frac{z}{1!} + + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + \cdots$$

The other solution is singular as $z \to 0$.

The only solution with the correct boundary condition at infinity,

$$U(z; a, b) \sim z^{-a}\left[1 + \mathcal{O}(z^{-1})\right] \quad \text{as} \quad z \to \infty,$$

was characterized by Tricomi (1947).

When $a = -k$ or $m = m_k = (2k-1)$, $k \in \mathbb{N}$, Kummer's power series $M(z; a, b)$ becomes a polynomial which connects to the Tricomi's function $U(z; a, b)$.

Two-scale asymptotic method

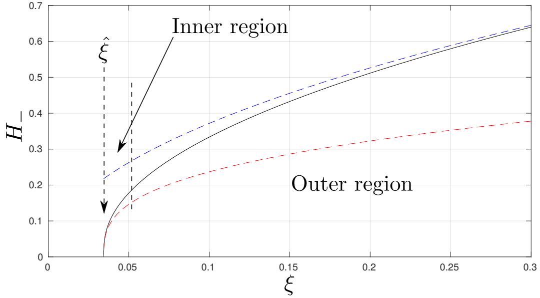


Figure: The numerical solution H_- (black solid line) for $m = 2.99$ and $n = 0$ with a schematic representation of the two asymptotic scales. The blue dashed line is the far-field behaviour and the red dashed line is the near field behaviour.

Outer region: Kummer's equation and power expansion

$$H_- = r^{\frac{2}{m+1}} + \alpha u_1(r) + \alpha^2 u_2(r) + \cdots$$

Inner region: Invariant manifold after blow-up technique.

Inner region ($n = 0$)

The *inner* solution near the interface

$$\phi = A + |A|^{\frac{m+1}{m-1}}\eta, \quad H(\phi) = |A|^{\frac{2}{m-1}}\mathcal{H}(\eta),$$

satisfies the differential equation

$$\frac{d}{d\eta}\left(\mathcal{H}^m \frac{d\mathcal{H}}{d\eta}\right) = 1 + \frac{m+1}{2}\operatorname{sign}(A)\frac{d\mathcal{H}}{d\eta} + |A|^{\frac{2}{m-1}}\left(\frac{m+1}{2}\eta\frac{d\mathcal{H}}{d\eta} - \mathcal{H}\right),$$

The inner solution is formally expanded as

$$\mathcal{H}(\eta) = \mathcal{H}_0(\eta) + |A|^{\frac{2}{m-1}}\mathcal{H}_1(\eta) + |A|^{\frac{4}{m-1}}\mathcal{H}_2(\eta) + \dots$$

where $\mathcal{H}_0$ satisfies after integration

$$\mathcal{H}_0^m \frac{d\mathcal{H}_0}{d\eta} = \eta + \frac{m+1}{2}\operatorname{sign}(A)\mathcal{H}_0.$$

Inner region $(n = 0)$

The first-order non-autonomous equation is equivalent to the planar system

$$\begin{cases} \dot{\eta} = \mathcal{H}_0^m, \\ \dot{\mathcal{H}}_0 = \eta + \frac{m+1}{2}\operatorname{sign}(A)\mathcal{H}_0, \end{cases}$$

where $(0,0)$ is located at the intersection of the center curve

$$W_c(0,0) = \left\{ \eta = -\frac{m+1}{2}\operatorname{sign}(A)\mathcal{H}_0 + \mathcal{O}(\mathcal{H}_0^m), \quad \mathcal{H}_0 \in \mathbb{R} \right\}$$

and the stable or unstable curve depending on the sign of A.

If $A > 0$, we have

$$\mathcal{H}_0(\eta) \sim \left(\frac{m(m+1)}{2}\eta \right)^{\frac{1}{m}} \quad \text{as} \quad \eta \to 0$$

and if $A < 0$, we have

$$\mathcal{H}_0(\eta) \sim \frac{2}{m+1}\eta \quad \text{as} \quad \eta \to 0.$$

In the far field, we always have

$$\mathcal{H}_0(\eta) \sim \left(\frac{m+1}{2}\eta^2 \right)^{\frac{1}{m+1}} \quad \text{as} \quad \eta \to \infty.$$

Outer region

Substituting

$$\phi = A + |A|^{\frac{m+1}{m-1}}\eta, \quad H(\phi) = |A|^{\frac{2}{m-1}}\mathcal{H}(\eta)$$

into

$$\mathcal{H}(\eta) = \mathcal{H}_0(\eta) + |A|^{\frac{2}{m-1}}\mathcal{H}_1(\eta) + |A|^{\frac{4}{m-1}}\mathcal{H}_2(\eta) + \ldots$$

and expanding in A brings the asymptotic expansion to the form:

$$H(\phi) \sim r^{\frac{2}{m+1}} - \frac{2A}{(m+1)x_*} r^{\frac{1-m}{1+m}} + \frac{(1-m)A^2}{2(m+1)} r^{-\frac{2m}{m+1}}.$$

This expansion is compared to the *outer* expansion:

$$H_- = r^{\frac{2}{m+1}} + \alpha u_1(r) + \alpha^2 u_2(r) + \cdots$$

with $r := \phi/x_*$ and Kummer's differential equation for u_1.

Matching conditions define α in terms of A and $m - m_k$ in terms of A. Behaviour at infinity defined $x_0 - x_*$ in terms of A.

Numerical confirmation: m_2

No bifurcation occurs at $m_1 = 1$. Bifurcation at $m_2 = 3$ and $n = 0$:

$$A = 3(x_0 - x_*) + \mathcal{O}((x_0 - x_*)^2)$$

and

$$3 - m = \mathcal{O}(A^2).$$

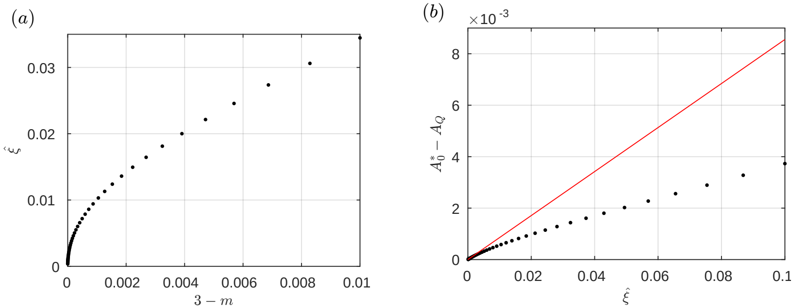


Figure: Panel (a) shows the variation of A versus $3 - m$ and panel (b) shows the variation of $x_0 - x_*$ versus A local to $m = 3$.

Numerical confirmation: m_3

Bifurcation at $m_3 = 5$ and $n = 0$:

$$A = -\frac{40}{9}(x_0 - x_*) + \cdots$$

and

$$5 - m = \frac{27\sqrt{3}}{4}A + \cdots$$

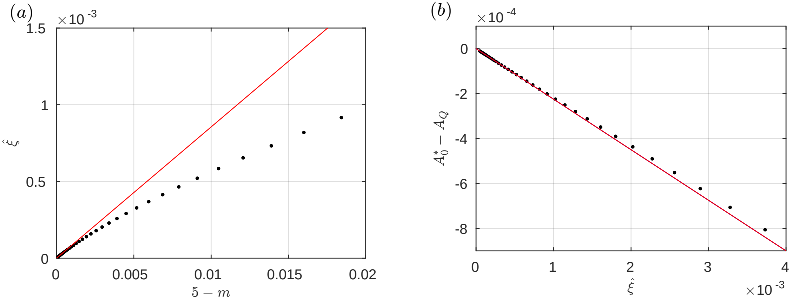


Figure: Panel (a) shows the variation of A versus $5 - m$ and panel (b) shows the variation of $x_0 - x_*$ versus A local to $m = 5$.

Numerical confirmation: m_4

Bifurcation at $m_4 = 7$ and $n = 0$:

$$A = \frac{105}{16}(x_0 - x_*) + \cdots$$

and

$$7 - m = \frac{2048}{15}A + \cdots$$

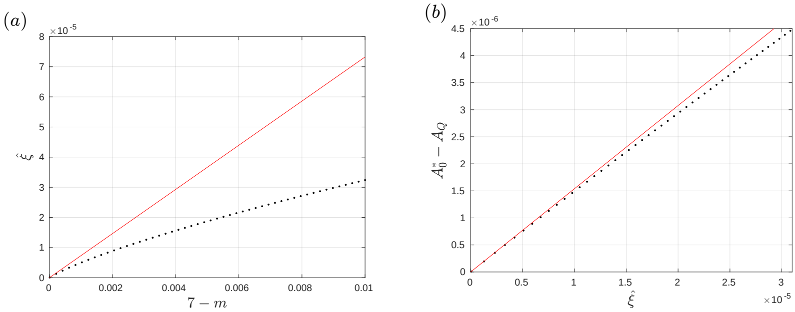


Figure: Panel (a) shows the variation of A versus $7 - m$ and panel (b) shows the variation of $x_0 - x_*$ versus A local to $m = 7$.

Conclusion

- For every $m > 0$, $n < 1$ and $m + n > 1$ a pair of solutions H_+ and H_- can be constructed numerically and then converted to $h(x, t)$
  - Solutions with $A_\pm > 0$ correspond to reversing interfaces
  - Solutions with $A_\pm < 0$ correspond to anti-reversing interfaces

- The behaviour of the self-similar solution at zero and infinity is justified by the dynamical system theory.

- Bifurcations of self-similar solutions are predicted from analysis of the classical Kummer's differential equation.

References

1. J. M. Foster, C. P. Please, A. D. Fitt, and G. Richardson, *The reversing of interfaces in slow diffusion processes with strong absorption,* SIAM J. Appl. Math. **72** (2012), 144–162
2. J. M. Foster and D. E. Pelinovsky, *Self-similar solutions for reversing interfaces in the slow diffusion equation with strong absorption,* SIAM J. Appl. Dynam. Syst. **15** (2016), 2017–2050.
3. J. M. Foster, P. Gysberg, J.R. King, and D. E. Pelinovsky, *Bifurcations of self-similar solutions for reversing interfaces in the slow diffusion equation,* Nonlinearity (2018), submitted.