Rogue waves on the periodic background

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The rogue wave of the cubic NLS equation

The focusing nonlinear Schrödinger (NLS) equation

$$i\psi_t + \frac{1}{2}\psi_{xx} + |\psi|^2\psi = 0$$

admits the exact solution

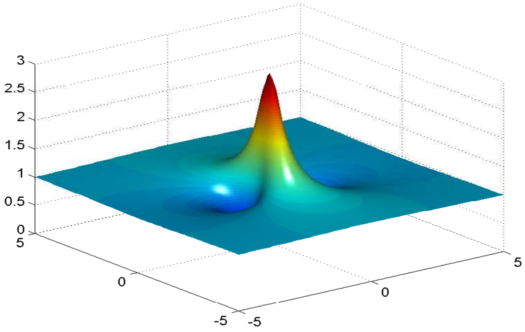
$$\psi(x,t) = \left[1 - \frac{4(1+2it)}{1+4x^2+4t^2}\right] e^{it}.$$

It was discovered by H. Peregrine (1983) and was labeled as *the rogue wave*.

Properties of the rogue wave:

- It is related to modulational instability of CW background $\psi_0(x,t) = e^{it}$.
- It comes from nowhere: $|\psi(x,t)| \to 1$ as $|x| + |t| \to \infty$.
- It is magnified at the center: $M_0 := |\psi(0,0)| = 3$.

The rogue wave of the cubic NLS equation



Possible developments:

- To generate higher-order rational solutions for multiple rogue waves...
- To extend constructions in other basic integrable PDEs...

Periodic wave background

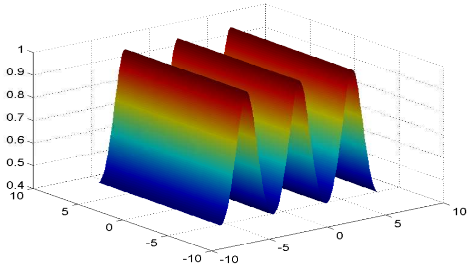
The focusing nonlinear Schrödinger (NLS) equation

$$i\psi_t + \frac{1}{2}\psi_{xx} + |\psi|^2\psi = 0$$

admits other wave solutions, e.g. the periodic waves of trivial phase

$$\psi_{\rm dn}(x,t) = {\rm dn}(x;k)e^{i(1-k^2/2)t}, \quad \psi_{\rm cn}(x,t) = k{\rm cn}(x;k)e^{i(k^2-1/2)t}$$

where $k \in (0,1)$ is elliptic modulus.



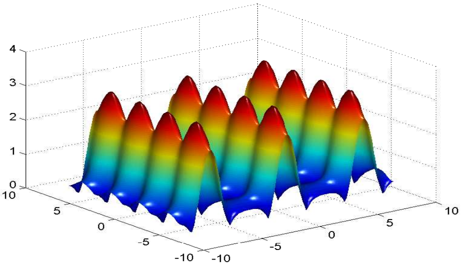
Double-periodic wave background

Double-periodic solutions (Akhmediev, Eleonskii, Kulagin, 1987):

$$\psi(x,t) = k\frac{\mathrm{cn}(t;k)\mathrm{cn}(\sqrt{1+k}x;\kappa) + i\sqrt{1+k}\mathrm{sn}(t;k)\mathrm{dn}(\sqrt{1+k}x;\kappa)}{\sqrt{1+k}\mathrm{dn}(\sqrt{1+k}x;\kappa) - \mathrm{dn}(t;k)\mathrm{cn}(\sqrt{1+k}x;\kappa)} e^{it},$$

$$\psi(x,t) = \frac{\mathrm{dn}(t;k)\mathrm{cn}(\sqrt{2}x;\kappa) + i\sqrt{k(1+k)}\mathrm{sn}(t;k)}{\sqrt{1+k} - \sqrt{k}\mathrm{cn}(t;k)\mathrm{cn}(\sqrt{2}x;\kappa)} e^{ikt}, \quad \kappa = \frac{\sqrt{1-k}}{\sqrt{2}}.$$

where $k \in (0,1)$ is elliptic modulus.



Main question

Can we obtain a rogue wave on the background ψ_0 such that

$$\inf_{x_0, t_0, \alpha_0 \in \mathbb{R}} \sup_{x \in \mathbb{R}} \left| \psi(x,t) - \psi_0(x - x_0, t - t_0) e^{i\alpha_0} \right| \to 0 \quad \text{as} \quad t \to \pm\infty \quad ???$$

This rogue wave *appears from nowhere and disappears without trace*.

Further questions:

- Magnification factors for rogue waves
- Spectral representation and inverse scattering
- Robustness (stability) in the time evolution.
- Extensions to quasi-periodic background.
- Extensions to multi-soliton background.

Darboux transformation as the main tool

Let u be a solution of the NLS. It is a potential of the compatible Lax system

$$\varphi_x = U(\lambda, u)\varphi, \qquad U(\lambda, u) = \begin{pmatrix} \lambda & u \\ -\bar{u} & -\lambda \end{pmatrix}$$

and

$$\varphi_t = V(\lambda, u)\varphi, \qquad V(\lambda, u) = i \begin{pmatrix} \lambda^2 + \frac{1}{2}|u|^2 & \frac{1}{2}u_x + \lambda u \\ \frac{1}{2}\bar{u}_x - \lambda \bar{u} & -\lambda^2 - \frac{1}{2}|u|^2 \end{pmatrix},$$

so that $\varphi_{xt} = \varphi_{tx}$.

Let $\varphi = (p_1, q_1)$ be a nonzero solution of the Lax system for $\lambda = \lambda_1 \in \mathbb{C}$. The following one-fold Darboux transformation (DT):

$$\hat{u} = u + \frac{2(\lambda_1 + \bar{\lambda}_1)p_1\bar{q}_1}{|p_1|^2 + |q_1|^2},$$

provides another solution $\hat{u}$ of the same NLS equation.

Preliminary literature

- Numerical computations of eigenfunctions for DT on *dn*-, *cn*-, and double-periodic backgrounds:
  (Kedziora–Ankiewicz–Akhmediev, 2014) (Calini–Schober, 2017)
- Emergence of rogue waves in simulations of modulation instability of *dn*-periodic waves:
  (Agafontsev–Zakharov, 2016)
- Magnification factors of quasi-periodic solutions from analysis of Riemann's Theta functions:
  (Bertola–Tovbis, 2017) (Wright, 2019)
- Rogue waves from superpositions of nearly identical solitons:
  (Bilman–Buckingham, 2018) (Slunyaev, 2019)

Algebraic method - Step 1

Consider the spectral problem

$$\varphi_x = U(\lambda, u)\varphi, \qquad U(\lambda, u) = \begin{pmatrix} \lambda & u \\ -\bar{u} & -\lambda \end{pmatrix}$$

Fix $\lambda = \lambda_1 \in \mathbb{C}$ with $\varphi = (p_1, q_1) \in \mathbb{C}^2$ and set

$$\begin{cases} u = p_1^2 + \bar{q}_1^2, \\ \bar{u} = \bar{p}_1^2 + q_1^2. \end{cases}$$

The spectral problem becomes the Hamiltonian system of degree two generated by the Hamiltonian function

$$H = \lambda_1 p_1 q_1 + \bar{\lambda}_1 \bar{p}_1 \bar{q}_1 + \frac{1}{2}(p_1^2 + \bar{q}_1^2)(\bar{p}_1^2 + q_1^2).$$

The algebraic technique is called the “nonlinearization” of Lax pair (Cao–Geng, 1990) (Cao–Wu–Geng, 1999) (R.Zhou, 2009)

Hamiltonian system and constraints

The Hamiltonian system is integrable with two constants of motion:

$$\begin{aligned} H &= \lambda_1 p_1 q_1 + \bar{\lambda}_1 \bar{p}_1 \bar{q}_1 + \frac{1}{2}(p_1^2 + \bar{q}_1^2)(\bar{p}_1^2 + q_1^2), \\ F &= i(p_1 q_1 - \bar{p}_1 \bar{q}_1). \end{aligned}$$

The constraints between u and (p_1, q_1) are extended as:

$$\begin{aligned} u &= p_1^2 + \bar{q}_1^2, \\ \frac{du}{dx} + 2iFu &= 2(\lambda_1 p_1^2 - \bar{\lambda}_1 \bar{q}_1^2), \\ \frac{d^2u}{dx^2} + 2|u|^2 u + 2iF\frac{du}{dx} - 4Hu &= 4(\lambda_1^2 p_1^2 + \bar{\lambda}_1^2 \bar{q}_1^2). \end{aligned}$$

Compatible potentials $u(x)$ satisfy the closed second-order ODE:

$$u'' + 2|u|^2 u + 2icu' - 4bu = 0,$$

where $c := F + i(\lambda_1 - \bar{\lambda}_1)$ and $b := H + iF(\lambda_1 - \bar{\lambda}_1) + |\lambda_1|^2$.

Integrability of the Hamiltonian system

The Hamiltonian system is a compatibility condition of the Lax equation

$$\frac{d}{dx} W(\lambda) = U(\lambda, u) W(\lambda) - W(\lambda) U(\lambda, u),$$

where $U(\lambda, u)$ is the same as in the Lax system and

$$W(\lambda) = \left(\begin{array}{cc} W_{11}(\lambda) & W_{12}(\lambda) \\ \bar{W}_{12}(-\lambda) & -\bar{W}_{11}(-\lambda) \end{array} \right),$$

with

$$\begin{aligned} W_{11}(\lambda) &= 1 - \frac{p_1 q_1}{\lambda - \lambda_1} + \frac{\bar{p}_1 \bar{q}_1}{\lambda + \bar{\lambda}_1}, \\ W_{12}(\lambda) &= \frac{p_1^2}{\lambda - \lambda_1} + \frac{\bar{q}_1^2}{\lambda + \bar{\lambda}_1}. \end{aligned}$$

Simple algebra shows

$$W_{11}(\lambda) = \frac{\lambda^2 + ic\lambda + b + \frac{1}{2}|u|^2}{(\lambda - \lambda_1)(\lambda + \bar{\lambda}_1)}, \quad W_{12}(\lambda) = \frac{u\lambda + icu + \frac{1}{2}u'}{(\lambda - \lambda_1)(\lambda + \bar{\lambda}_1)}.$$

Closure relations

The $(1, 2)$-element of the Lax equation,

$$\frac{d}{dx} W_{12}(\lambda) = 2\lambda W_{12}(\lambda) - 2u W_{11}(\lambda),$$

yields the second-order equation on u:

$$u'' + 2|u|^2 u + 2icu' - 4bu = 0.$$

$\det W(\lambda)$ is constant in (x, t) and has simple poles at λ_1 and $-\bar{\lambda}_1$:

$$\det[W(\lambda)] = -[W_{11}(\lambda)]^2 - W_{12}(\lambda)\bar{W}_{12}(-\lambda) = -\frac{P(\lambda)}{(\lambda - \lambda_1)^2(\lambda + \bar{\lambda}_1)^2}$$

so that $P(\lambda)$ is constant in (x, t) and has roots at λ_1 and $-\bar{\lambda}_1$:

$$P(\lambda) = (\lambda^2 + ic\lambda + b + \frac{1}{2}|u|^2)^2 - (u\lambda + icu + \frac{1}{2}u')(\bar{u}\lambda + ic\bar{u} - \frac{1}{2}\bar{u}')$$

Conserved quantities

The second-order equation on u

$$u'' + 2|u|^2 u + 2icu' - 4bu = 0$$

is now closed with the conserved quantities

$$\left. \begin{array}{r} i(u'\bar{u} - u\bar{u}') - 2c|u|^2 = 4a, \\ |u'|^2 + |u|^4 + 4b|u|^2 = 8d. \end{array} \right\}$$

These equations describe a general class of **traveling wave solutions**:

$$\psi(x,t) = u(x+ct)e^{-2ibt}.$$

The polynomial $P(\lambda)$ in $\det W(\lambda)$ is given by

$$P(\lambda) = \lambda^4 + 2ic\lambda^3 + (2b - c^2)\lambda^2 + 2i(a + bc)\lambda + b^2 - 2ac + 2d,$$

with roots at λ_1 and $-\bar{\lambda}_1$. (Another pair also exists.)

Periodic waves of trivial phase

For traveling wave solutions:

- $c = 0$ can be set without loss of generality.
- $a = 0$ is set for waves with trivial phase.

The real function $u(x)$ is determined by the quadrature:

$$\left(\frac{du}{dx}\right)^2 + u^4 + 4bu^2 = 8d$$

with two parameters b,d. Parameterizing $V(u) = u^4 + 4bu^2 - 8d$ by two pairs of roots:

$$\left\{ \begin{array}{l} -4b = u_1^2 + u_2^2, \\ -8d = u_1^2 u_2^2. \end{array} \right.$$

we get **two families of traveling wave solutions:**

- $0 < u_2 < u_1$: $u(x) = u_1 \mathrm{dn}(u_1 x; k)$
- $u_2 = i\nu_2$: $u(x) = u_1 \mathrm{cn}(\alpha x; k)$, $\alpha = \sqrt{u_1^2 + \nu_2^2}$

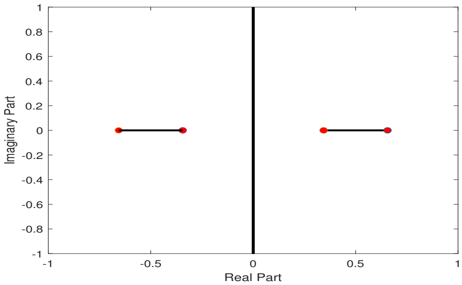
Lax spectrum of dn-periodic waves

Polynomial $P(\lambda)$ simplifies in terms of the turning points u_1, u_2:

$$P(\lambda) = \lambda^4 - \frac{1}{2}(u_1^2 + u_2^2)\lambda^2 + \frac{1}{16}(u_1^2 - u_2^2)^2$$

with two pairs of roots

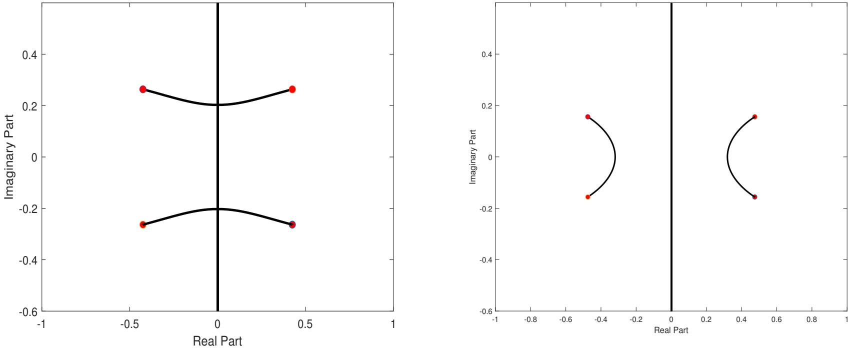
$$\lambda_1^{\pm} = \pm\frac{u_1 + u_2}{2}, \quad \lambda_2^{\pm} = \pm\frac{u_1 - u_2}{2}.$$



Lax spectrum of cn-periodic waves

If $u_2 = i\nu_2$, there is one quadruplet of roots:

$$\lambda_1^{\pm} = \pm \frac{u_1 + i\nu_2}{2}, \quad \lambda_2^{\pm} = \pm \frac{u_1 - i\nu_2}{2}.$$

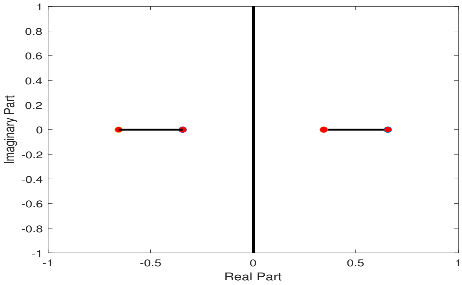


En route to rogue waves

Let $\varphi = (p_1, q_1)$ be a nonzero solution of the Lax system for $\lambda = \lambda_1 \in \mathbb{C}$. The one-fold Darboux transformation

$$\hat{u} = u + \frac{2(\lambda_1 + \bar{\lambda}_1) p_1 \bar{q}_1}{|p_1|^2 + |q_1|^2},$$

gives another solution $\hat{u}$ of the same NLS equation.



Question: which value of λ_1 to use?

Algebraic method - Step 2

Evaluating the matrix elements at simple poles λ_1 and $-\bar{\lambda}_1$

$$W_{11}(\lambda) = 1 - \frac{p_1 q_1}{\lambda - \lambda_1} + \frac{\bar{p}_1 \bar{q}_1}{\lambda + \bar{\lambda}_1} = \frac{\lambda^2 + ic\lambda + b + \frac{1}{2}|u|^2}{(\lambda - \lambda_1)(\lambda + \bar{\lambda}_1)},$$

$$W_{12}(\lambda) = \frac{p_1^2}{\lambda - \lambda_1} + \frac{\bar{q}_1^2}{\lambda + \bar{\lambda}_1} = \frac{u\lambda + icu + \frac{1}{2}u'}{(\lambda - \lambda_1)(\lambda + \bar{\lambda}_1)},$$

we can derive the inverse relations between the potential u and the squared eigenfunctions:

$$p_1^2 = \frac{1}{\lambda_1 + \bar{\lambda}_1}\left(\frac{1}{2}u' + icu + \lambda_1 u\right),$$

$$q_1^2 = \frac{1}{\lambda_1 + \bar{\lambda}_1}\left(-\frac{1}{2}u' + icu + \lambda_1 u\right),$$

$$p_1 q_1 = -\frac{1}{\lambda_1 + \bar{\lambda}_1}\left(b + \frac{1}{2}|u|^2 + i\lambda_1 c + \lambda_1^2\right).$$

The eigenfunction $\varphi = (p_1, q_1)$ is periodic if u is periodic.

Second linearly independent solution

Let us define the second solution $\varphi = (\hat{p}_1, \hat{q}_1)$ by

$$\hat{p}_1 = p_1 \phi_1 - \frac{2\bar{q}_1}{|p_1|^2 + |q_1|^2}, \quad \hat{q}_1 = q_1 \phi_1 + \frac{2\bar{p}_1}{|p_1|^2 + |q_1|^2},$$

such that $p_1 \hat{q}_1 - \hat{p}_1 q_1 = 2$ (Wronskian is constant). Then, scalar function $\phi_1(x,t)$ satisfies

$$\frac{\partial \phi_1}{\partial x} = -\frac{4(\lambda_1 + \bar{\lambda}_1)\bar{p}_1 \bar{q}_1}{(|p_1|^2 + |q_1|^2)^2}$$

and

$$\frac{\partial \phi_1}{\partial t} = -\frac{4i(\lambda_1^2 - \bar{\lambda}_1^2)\bar{p}_1 \bar{q}_1}{(|p_1|^2 + |q_1|^2)^2} + \frac{2i(\lambda_1 + \bar{\lambda}_1)(u\bar{p}_1^2 + \bar{u}\bar{q}_1^2)}{(|p_1|^2 + |q_1|^2)^2}.$$

The system is compatible as it is obtained from Lax equation.

Second solutions for periodic waves

For periodic waves with the trivial phase, variables are separated by

$$u(x,t) = U(x)e^{-2ibt}, \quad p_1(x,t) = P_1(x)e^{-ibt}, \quad q_1(x,t) = Q_1(x)e^{ibt},$$

where U is real, either $U(x) = \mathrm{dn}(x;k)$ or $U(x) = k\,\mathrm{cn}(x;k)$, whereas $|p_1|^2 + |q_1|^2 = \mathrm{dn}(x;k)$ in both cases.

Integrating linear equations for $\phi_1(x,t)$ yields

$$\phi_1(x,t) = 2x + 2i(1 \pm \sqrt{1-k^2})t \pm 2\sqrt{1-k^2}\int_0^x \frac{dy}{\mathrm{dn}^2(y;k)}$$

and

$$\phi_1(x,t) = 2k^2 \int_0^x \frac{\mathrm{cn}^2(y;k)dy}{\mathrm{dn}^2(y;k)} \mp 2ik\sqrt{1-k^2}\int_0^x \frac{dy}{\mathrm{dn}^2(y;k)} + 2ikt$$

from which it is obvious that $|\phi_1| \to \infty$ as $t \to \pm\infty$.

Algebraic method - Step 3

Rogue waves on the background u are generated by the DT:

$$\hat{u} = u + \frac{2(\lambda_1 + \bar{\lambda}_1)\hat{p}_1\bar{\hat{q}}_1}{|\hat{p}_1|^2 + |\hat{q}_1|^2},$$

where

$$\hat{p}_1 = p_1\phi_1 - \frac{2\bar{q}_1}{|p_1|^2 + |q_1|^2}, \quad \hat{q}_1 = q_1\phi_1 + \frac{2\bar{p}_1}{|p_1|^2 + |q_1|^2},$$

As $t \to \pm\infty$,

$$\hat{u}(x,t)|_{|\phi_1|\to\infty} = u + \frac{2(\lambda_1 + \bar{\lambda}_1)p_1\bar{q}_1}{|p_1|^2 + |q_1|^2}$$

which is a translation of the periodic wave u, e.g.

$$\hat{u}(x,t)|_{|\phi_1|\to\infty} = \frac{\sqrt{1-k^2}}{\mathrm{dn}(x;k)} = \mathrm{dn}(x + K(k);k)$$

or

$$\hat{u}(x,t)|_{|\phi_1|\to\infty} = -\frac{k\sqrt{1-k^2}\mathrm{sn}(x;k)}{\mathrm{dn}(x;k)} = k\mathrm{cn}(x + K(k);k).$$

Magnification factor

Rogue waves on the background u are generated by the DT:

$$\hat{u} = u + \frac{2(\lambda_1 + \bar{\lambda}_1)\hat{p}_1 \bar{\hat{q}}_1}{|\hat{p}_1|^2 + |\hat{q}_1|^2},$$

where

$$\hat{p}_1 = p_1 \phi_1 - \frac{2\bar{q}_1}{|p_1|^2 + |q_1|^2}, \quad \hat{q}_1 = q_1 \phi_1 + \frac{2\bar{p}_1}{|p_1|^2 + |q_1|^2},$$

At the center of the rogue wave,

$$\hat{u}(x,t)|_{\phi_1=0} = u - \frac{2(\lambda_1 + \bar{\lambda}_1)p_1 \bar{q}_1}{|p_1|^2 + |q_1|^2} = 2u - \tilde{u},$$

hence the magnification factor does not exceed *three* in the one-fold transformation.

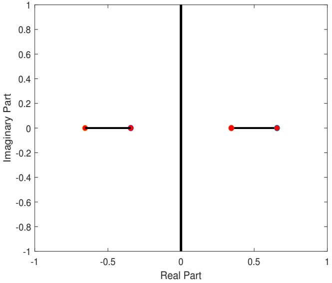
Rogue wave on the *dn*-periodic wave

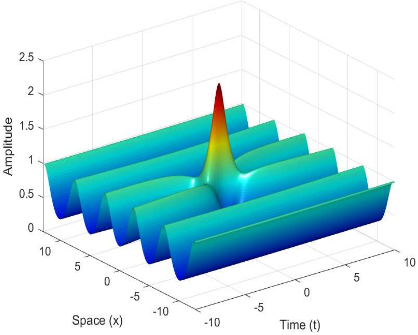
The *dn*-periodic wave is

$$u(x,t) = \mathrm{dn}(x;k)e^{i(1-k^2/2)t}$$

The rogue wave for the larger eigenvalue λ_1 has the larger magnification:

$$M(k) = 2 + \sqrt{1-k^2}, \quad k \in [0,1].$$





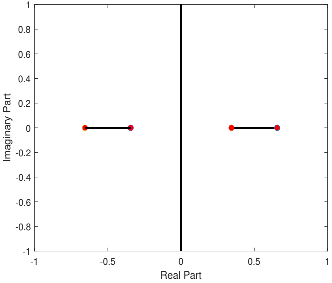
Another rogue wave on the *dn*-periodic wave

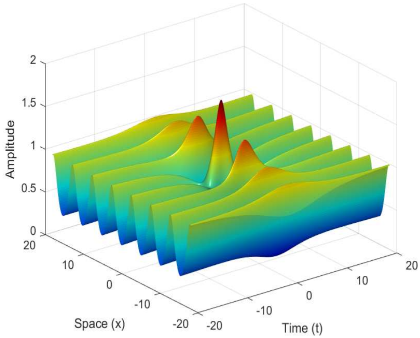
The *dn*-periodic wave is

$$u(x,t) = \mathrm{dn}(x;k)e^{i(1-k^2/2)t}$$

The rogue wave for the smaller eigenvalue λ_1 has the smaller magnification:

$$M(k) = 2 - \sqrt{1-k^2}, \quad k \in [0,1].$$





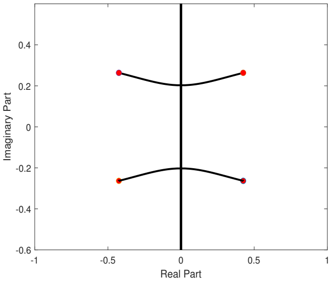
Rogue wave on the *cn*-periodic wave

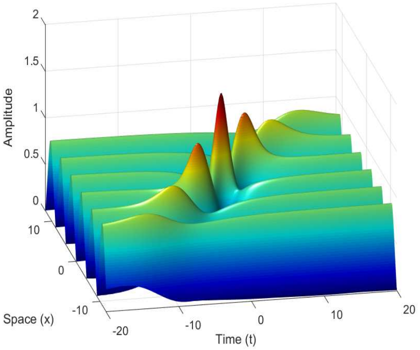
The *cn*-periodic wave is

$$\psi_{\rm cn}(x,t) = k{\rm cn}(x;k)e^{i(k^2-1/2)t}$$

The rogue wave has the exact magnification factor:

$$M(k) = 2, \quad k \in [0,1].$$





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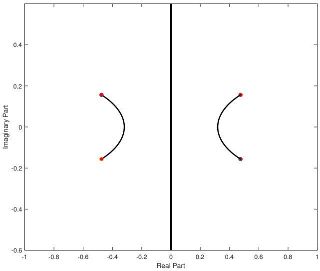
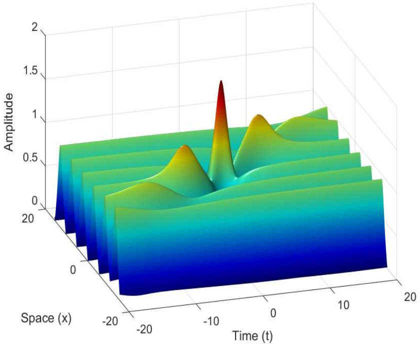


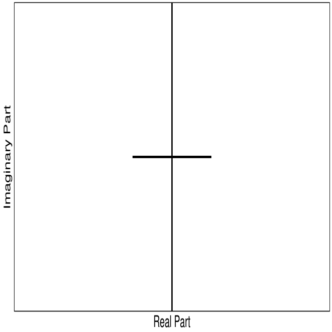


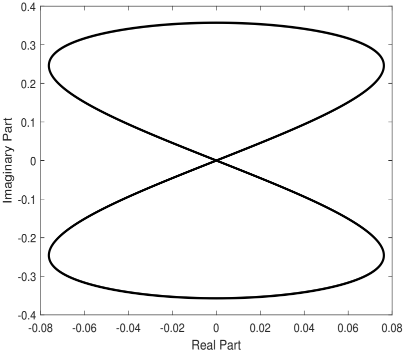
Relation to modulation instability of the periodic wave

If λ belongs to the Lax spectrum and $P(\lambda)$ is the polynomial in

$$P(\lambda) = \lambda^4 - \frac{1}{2}(u_1^2 + u_2^2)\lambda^2 + \frac{1}{16}(u_1^2 - u_2^2)^2$$

then $\Gamma := \pm 2i\sqrt{P(\lambda)}$ is in the modulation instability spectrum.
(Deconinck–Segal, 2017) (Deconinck–Upsal, 2019)





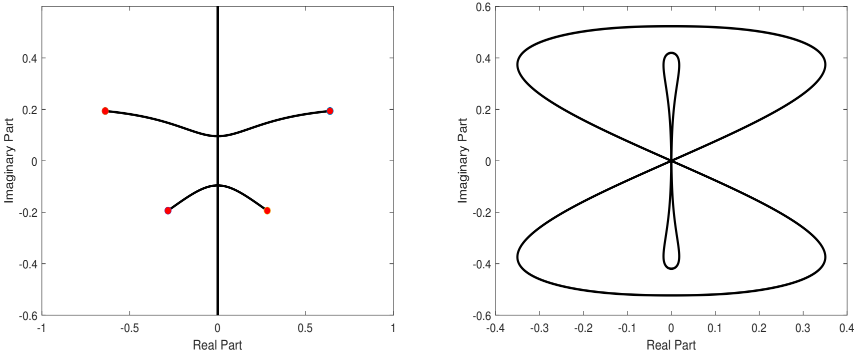
Relation to modulation instability of the periodic wave

Here is an example of the periodic wave with nontrivial phase

$$u(x) = R(x) e^{i\Theta(x)} e^{2ibt}$$

with

$$R(x) = \sqrt{\beta - k^2 \mathrm{sn}^2(x;k)}, \quad \Theta(x) = -2e \int_0^x \frac{dx}{R(x)^2}.$$



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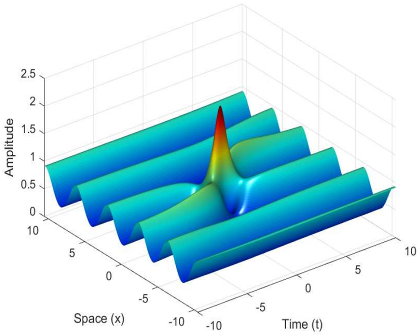
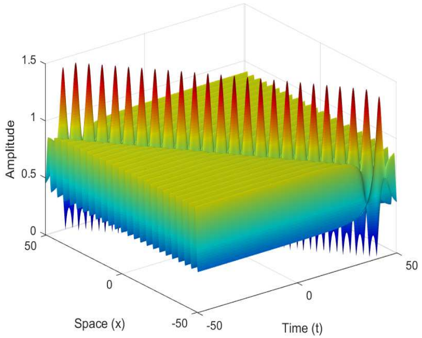




Algebraic method with two eigenvalues

Fix $\lambda = \lambda_1 \in \mathbb{C}$ with $\varphi = (p_1, q_1) \in \mathbb{C}^2$ and $\lambda = \lambda_2 \in \mathbb{C}$ with $\varphi = (p_2, q_2) \in \mathbb{C}^2$ such that $\lambda_1 \neq \pm\lambda_2$ and $\lambda_1 \neq \pm\bar{\lambda}_2$. Set

$$u = p_1^2 + \bar{q}_1^2 + p_2^2 + \bar{q}_2^2.$$

The algebraic method produces the third-order equation

$$u''' + 6|u|^2 u' + 2ic(u'' + 2|u|^2 u) + 4bu' + 8iau = 0,$$

with three constants of motion:

$$\left.\begin{array}{r} d + \frac{1}{2}b|u|^2 + \frac{i}{4}c(u'\bar{u} - u\bar{u}') + \frac{1}{8}(u\bar{u}'' + u''\bar{u} - |u'|^2 + 3|u|^4) = 0, \\ 2e - a|u|^2 - \frac{1}{4}c(|u'|^2 + |u|^4) + \frac{i}{8}(u''\bar{u}' - u'\bar{u}'') = 0, \\ f - \frac{i}{2}a(u'\bar{u} - u\bar{u}') + \frac{1}{4}b(|u'|^2 + |u|^4) + \frac{1}{16}(|u'' + 2|u|^2 u|^2 - (u'\bar{u} - u\bar{u}')^2) = 0. \end{array}\right\}$$

Eigenvalues λ_1 and λ_2 are found among three roots of the polynomial

$$\begin{aligned} P(\lambda) &= \lambda^6 + 2ic\lambda^5 + (2b - c^2)\lambda^4 + 2i(a + bc)\lambda^3 + (b^2 - 2ac + 2d)\lambda^2 \\ &\quad + 2i(e + ab + cd)\lambda + f + 2bd - 2ce - a^2. \end{aligned}$$

Double-periodic solutions

Double-periodic solutions (Akhmediev, Eleonskii, Kulagin, 1987) correspond to $c = a = e = 0$. The solution takes the explicit form:

$$u(x,t) = [Q(x,t) + i\delta(t)]\, e^{i\theta(t)}.$$

where $Q(x,t)$ and $\delta(t)$ are found from the first-order quadratures:

$$\delta(t) = \frac{\sqrt{z_1 z_3}\,\mathrm{sn}(\mu t; k)}{\sqrt{z_3 - z_1 \mathrm{cn}^2(\mu t; k)}},$$

with $0 \leq z_1 \leq z_2 \leq z_3$ and

$$Q(x,t) = Q_4 + \frac{(Q_1 - Q_4)(Q_2 - Q_4)}{(Q_2 - Q_4) + (Q_1 - Q_2)\mathrm{sn}^2(\nu x; \kappa)},$$

with $Q_4 \leq Q_3 \leq Q_2 \leq Q_1$.

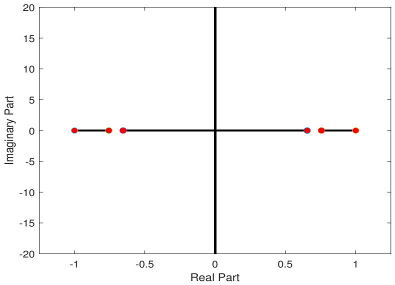
By construction, $\pm\sqrt{z_1}$, $\pm\sqrt{z_2}$, $\pm\sqrt{z_3}$ are roots of $P(\lambda)$:

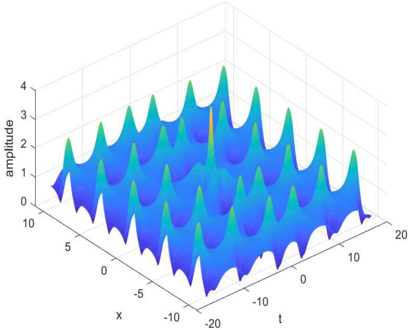
$$P(\lambda) = \lambda^6 + 2b\lambda^4 + (b^2 + 2d)\lambda^2 + f + 2bd.$$

Lax spectrum and rogue waves

The double-periodic solution if $z_{1,2,3}$ are real:

$$u(x,t) = k\frac{\mathrm{cn}(t;k)\mathrm{cn}(\sqrt{1+k}x;\kappa) + i\sqrt{1+k}\mathrm{sn}(t;k)\mathrm{dn}(\sqrt{1+k}x;\kappa)}{\sqrt{1+k}\mathrm{dn}(\sqrt{1+k}x;\kappa) - \mathrm{dn}(t;k)\mathrm{cn}(\sqrt{1+k}x;\kappa)} e^{it},$$

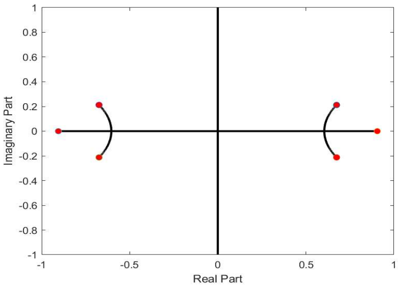


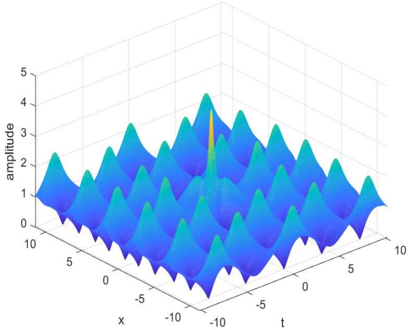


Lax spectrum and rogue waves

The double-periodic solution if z_1 is real and $z_{2,3}$ are complex:

$$u(x,t) = \frac{\mathrm{dn}(t;k)\mathrm{cn}(\sqrt{2}x;\kappa) + i\sqrt{k(1+k)}\mathrm{sn}(t;k)}{\sqrt{1+k} - \sqrt{k}\mathrm{cn}(t;k)\mathrm{cn}(\sqrt{2}x;\kappa)} e^{ikt}, \quad \kappa = \frac{\sqrt{1-k}}{\sqrt{2}}.$$





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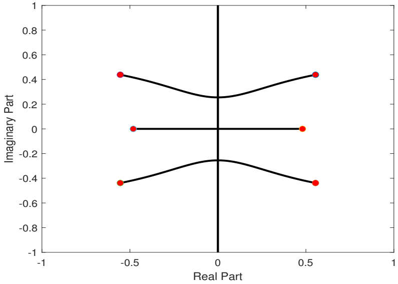

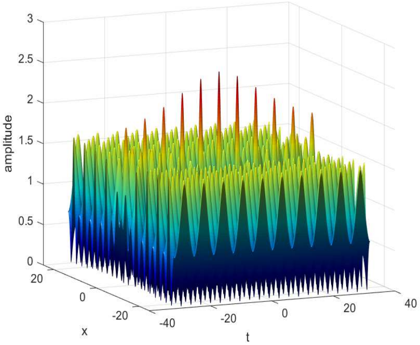

Summary

Summary:

- New method is developed for computations of eigenvalues and eigenfunctions of the Lax system for periodic and double-periodic waves.
- New exact solutions are obtained for rogue waves on the background of periodic and double-periodic waves.
- Magnification factor is computed exactly at the rogue waves.

Further directions:

- Characterize eigenvalues, eigenfunctions, and rogue waves on general quasi-periodic solutions.
- Observe rogue waves on the periodic background in water wave experiments (Amin Chabchoub, Sydney).

Thank you! Questions???

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