Persistence and stability of discrete vortices

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$$
\begin{aligned}
i\dot{u}_{n,m} &+ \epsilon \left(u_{n+1,m} + u_{n-1,m} + u_{n,m+1} + u_{n,m-1} - 4u_{n,m} \right) \\
&+ |u_{n,m}|^2 u_{n,m} = 0, \qquad\qquad (n,m) \in \mathbb{Z}^2
\end{aligned}
$$

Joint work with P. Kevrekidis (University of Massachusetts)

"Discrete and Continuous Models in Nonlinear Optics"

SIAM Conference on Dynamical Systems, May 26, 2005

Experimental motivations

- Bose-Einstein condensates in optical lattices
- Light-induced photonic lattices
- Coupled optical waveguides

Persistence of localized solutions

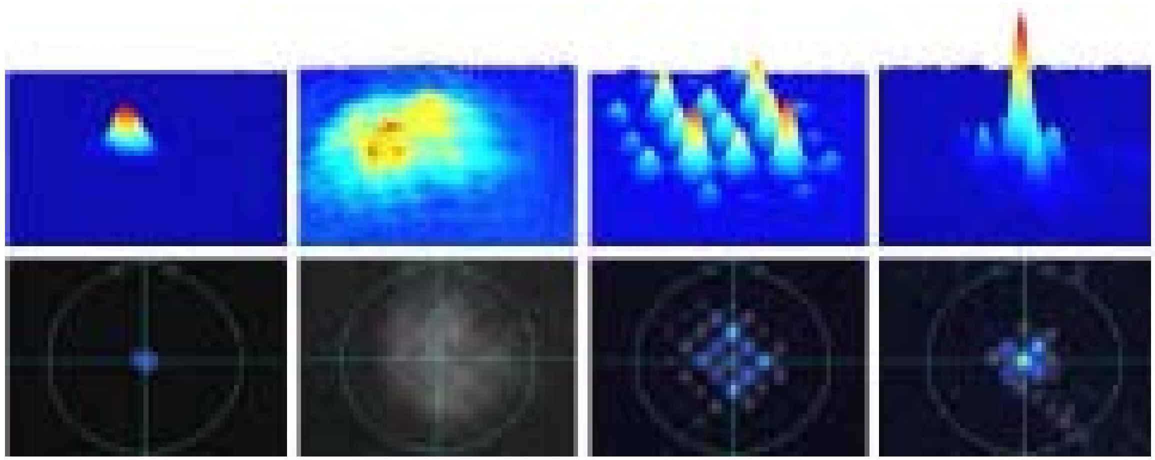
- Implicit Function Theorem
- Lyapunov–Schmidt reductions

Stability of localized solutions

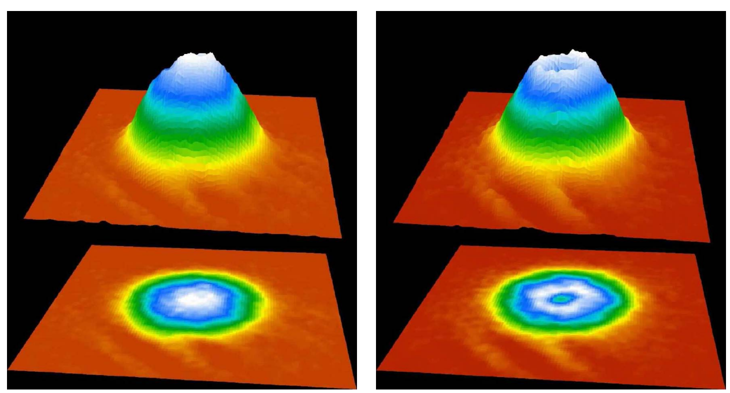
- Splitting of zero eigenvalues
- Negative index theory

Experimental pictures

- Discrete solitons



- Discrete vortices



Main Formalism

$$i\dot{u}_{n,m} + \epsilon\left(u_{n+1,m} + u_{n-1,m} + u_{n,m+1} + u_{n,m-1} - 4u_{n,m}\right) + |u_{n,m}|^2 u_{n,m} = 0, \qquad (n,m) \in \mathbb{Z}^2$$

- Vector space $\Omega = l^2(\mathbb{Z}^2, \mathbb{C})$ for $\{u_{n,m}\}_{(n,m)\in\mathbb{Z}^2}$:

$$(\mathbf{u}, \mathbf{w})_\Omega = \sum_{(n,m)\in\mathbb{Z}^2} \bar{u}_{n,m} w_{n,m}$$

- Hamiltonian formulation:

$$i\dot{u}_{n,m} = \frac{\partial H}{\partial \bar{u}_{n,m}},$$

where

$$H = \sum_{(n,m)\in\mathbb{Z}^2} \epsilon|u_{n+1,m} - u_{n,m}|^2 + |u_{n,m+1} - u_{n,m}|^2 - \frac{1}{2}|u_{n,m}|^4$$

- Existence problem for time-periodic localized solutions

$$u_{n,m}(t) = \phi_{n,m} e^{i(1-4\epsilon)t + i\theta_0}, \qquad \theta_0 \in \mathbb{R}$$

such that

$$(1 - |\phi_{n,m}|^2)\phi_{n,m} = \epsilon \left(\phi_{n+1,m} + \phi_{n-1,m} + \phi_{n,m+1} + \phi_{n,m-1}\right).$$

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- Stability problem for time-periodic localized solutions

$$u_{n,m}(t) = e^{i(1-4\epsilon)t + i\theta_0} \left(\phi_{n,m} + a_{n,m} e^{\lambda t} + \bar{b}_{n,m} e^{\bar{\lambda} t} \right)$$

such that

$$\left(1 - 2|\phi_{n,m}|^2\right) a_{n,m} - \phi_{n,m}^2 b_{n,m} - \epsilon \left(a_{n+1,m} + a_{n-1,m} + a_{n,m+1} + a_{n,m-1} \right) = i\lambda a_{n,m}$$

$$-\bar{\phi}_{n,m}^2 a_{n,m} + \left(1 - 2|\phi_{n,m}|^2\right) b_{n,m} - \epsilon \left(b_{n+1,m} + b_{n-1,m} + b_{n,m+1} + b_{n,m-1} \right) = -i\lambda b_{n,m}$$

where $\lambda \in \mathbb{C}$ and $(\mathbf{a}, \mathbf{b}) \in \Omega \times \Omega$

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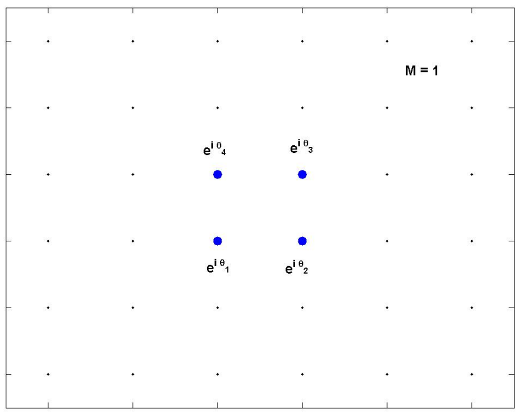
- Time-dependent nonlinear dynamics of localized solutions

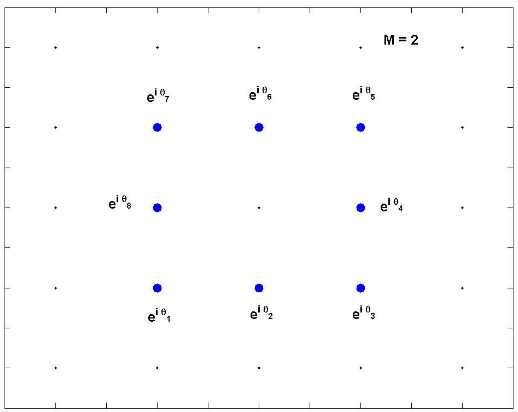
Existence problem

$$(1 - |\phi_{n,m}|^2)\phi_{n,m} = \epsilon \left(\phi_{n+1,m} + \phi_{n-1,m} + \phi_{n,m+1} + \phi_{n,m-1}\right)$$

Limiting solution:

$$\epsilon = 0: \qquad \phi^{(0)}_{n,m} = \begin{cases} e^{i\theta_{n,m}}, & (n,m) \in S, \\ 0, & (n,m) \in \mathbb{Z}^2 \backslash S, \end{cases}$$





Examples of a square discrete contour S

Existence problem

$$(1-|\phi_{n,m}|^2)\phi_{n,m} = \epsilon\left(\phi_{n+1,m}+\phi_{n-1,m}+\phi_{n,m+1}+\phi_{n,m-1}\right)$$

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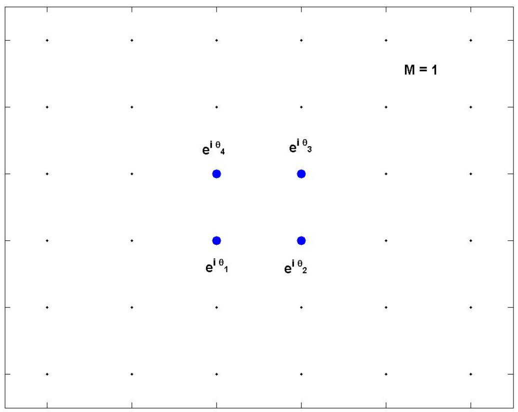

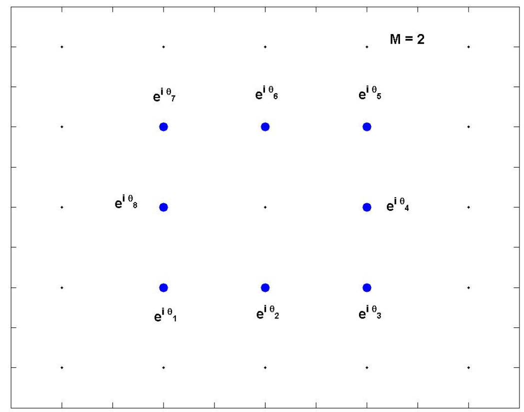

Examples of a square discrete contour S

What phase configurations $\theta_{n,m}$ can be continued for $\epsilon \neq 0$?

Lyapunov-Schmidt reductions

Proposition: Let $N = \dim(S)$ and $\mathcal{T}$ be the torus on $[0, 2\pi]^N$. There exists a vector-valued function $\mathbf{g} : \mathcal{T} \mapsto \mathbb{R}^N$, such that the limiting solution is continued to $\epsilon \neq 0$ if and only if $\boldsymbol{\theta} \in \mathcal{T}$ is a root of $\mathbf{g}(\boldsymbol{\theta}, \epsilon) = \mathbf{0}$.

- The Jacobian of the nonlinear system:

$$\mathcal{H} = \begin{pmatrix} 1 - 2|\phi_{n,m}|^2 & -\phi_{n,m}^2 \\ -\bar{\phi}_{n,m}^2 & 1 - 2|\phi_{n,m}|^2 \end{pmatrix} - \epsilon \delta_{\pm 1, \pm 1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- $\mathcal{H}$ is a self-adjoint Fredholm operator of index zero:

$$\dim(\ker(\mathcal{H}^{(0)}) = N$$

- Analytic functions:

$$\mathbf{g}(\boldsymbol{\theta}, \epsilon) = \sum_{k=1}^{\infty} \epsilon^k \mathbf{g}^{(k)}(\boldsymbol{\theta})$$

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- Let $\boldsymbol{\theta}_*$ be the root of $\mathbf{g}^{(1)}(\boldsymbol{\theta}) = \mathbf{0}$ and $\mathcal{M}_1 = \mathcal{D}\mathbf{g}^{(1)}(\boldsymbol{\theta}_*)$.
  If $\dim(\ker(\mathcal{M}_1)) = 1$, there exists a unique continuation of the limiting solution for $\epsilon \neq 0$.

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- Let $\boldsymbol{\theta}_*$ be a $(1+d)$-parameter solution of $\mathbf{g}^{(1)}(\boldsymbol{\theta}) = \mathbf{0}$. The limiting solution can not be continued to $\epsilon \neq 0$ if $\mathbf{g}^{(2)}(\boldsymbol{\theta}_*)$ is not orthogonal to $\ker(\mathcal{M}_1)$.

First-order reductions : classification of solutions

$$\mathbf{g}_j^{(1)}(\boldsymbol{\theta}) = \sin(\theta_j - \theta_{j+1}) + \sin(\theta_j - \theta_{j-1}) = 0, \quad 1 \leq j \leq 4M$$

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- (1) Discrete solitons

$$\theta_j = \{0, \pi\}, \qquad 1 \le j \le 4M$$

- (2) Symmetric vortices of charge L

$$\theta_j = \frac{\pi L(j-1)}{2M}, \qquad 1 \le j \le 4M,$$

- (3) One-parameter asymmetric vortices of charge $L = M$

$$\theta_{j+1} - \theta_j = \left\{ \begin{array}{c} \theta \\ \pi - \theta \end{array} \right\} \mod(2\pi), \qquad 1 \le j \le 4M$$

where

- M is number of nodes at each side of the square contour
- L is the vortex charge (winding number)

First-order reductions : persistence of solutions

$$\mathcal{M}_1 = \begin{pmatrix} a_1 + a_2 & -a_2 & 0 & \dots & a_1 \\ -a_2 & a_2 + a_3 & -a_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ -a_1 & 0 & 0 & \dots & a_{N-1} + a_N \end{pmatrix}, \quad a_j = \cos(\theta_{j+1} - \theta_j)$$

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- $\mathcal{M}_1$ has a simple zero eigenvalue if all $a_j \neq 0$ and

$$\left(\prod_{i=1}^{N} a_i \right) \left(\sum_{i=1}^{N} \frac{1}{a_i} \right) \neq 0.$$

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Family (1) persists for $\epsilon \neq 0$

- If all $a_j = a = \cos(\frac{\pi L}{2M})$, eigenvalues of $\mathcal{M}_1$ are:
$$\lambda_n = 4a \sin^2 \frac{\pi n}{4M}, \qquad 1 \leq n \leq 4M$$
Family (2) persists for $\epsilon \neq 0$ and $L \neq M$

Second-order reductions : termination of solutions

- If all $a_j = \pm a = \cos\theta$, there are $2M-1$ negative eigenvalues of $\mathcal{M}_1$, 2 zero eigenvalues and $2M-1$ positive eigenvalues of $\mathcal{M}_1$.

- Persistence of family (3) depends on $\mathbf{g}^{(2)}(\boldsymbol{\theta})$

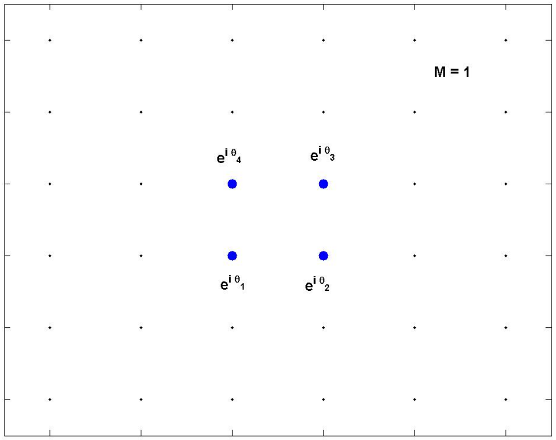
$$\mathbf{g}_j^{(2)} = \frac{1}{2}\sin(\theta_{j+1}-\theta_j)\left[\cos(\theta_j-\theta_{j+1})+\cos(\theta_{j+2}-\theta_{j+1})\right] + \frac{1}{2}\sin(\theta_{j-1}-\theta_j)\left[\cos(\theta_j-\theta_{j-1})+\cos(\theta_{j-2}-\theta_{j-1})\right]$$

- If $\ker(\mathcal{M}_1) = \{\mathbf{p}_0, \mathbf{p}_1\}$, then $(\mathbf{g}^{(2)}, \mathbf{p}_1) \neq 0$.

- Family (3) terminates except for one symmetric configuration:

$$\theta_1 = 0, \quad \theta_2 = \theta, \qquad \theta_3 = \pi, \quad \theta_4 = \pi + \theta,$$

Higher-order reductions : termination of the last family



- Symbolic software algorithm is used on a squared domain of N_0-by-N_0 lattice nodes, where $N_0 = 2K + 2M + 1$, and K is the order of the Lyapunov-Schmidt reductions.

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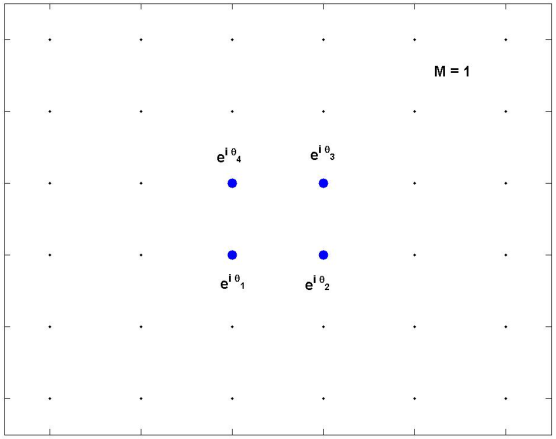

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- Super-symmetric family (3) has $\mathbf{g}^{(k)}(\boldsymbol{\theta}) = 0$ for $k = 1, 2, 3, 4, 5$ but $\mathbf{g}^{(6)}(\boldsymbol{\theta}) \neq 0$, unless $\theta_{j+1} - \theta_j = \frac{\pi}{2}$.
- Moreover, $(\mathbf{g}^{(6)}, \mathbf{p}_1) \neq 0$.

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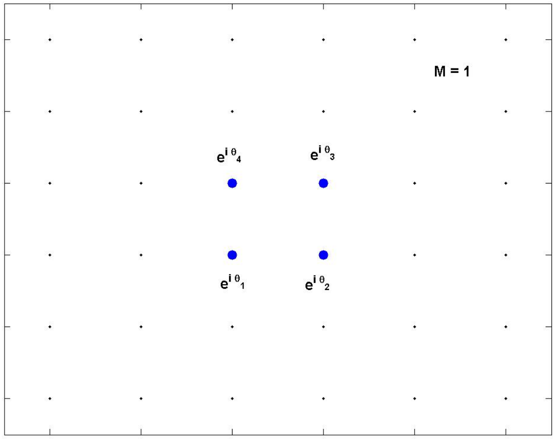

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- Moreover, $(\mathbf{g}^{(6)}, \mathbf{p}_1) \neq 0$.
- All asymmetric vortices (3) terminate

Zero eigenvalues of the stability problem

- Matrix-vector Hamiltonian form of the stability problem:

$$\mathcal{H}\boldsymbol{\psi} = i\lambda\sigma\boldsymbol{\psi},$$

  where
  - $\boldsymbol{\psi} \in l^2(\mathbb{Z}^2, \mathbb{C}^2)$
  - $\mathcal{H}$ is the Jacobian (energy) operator
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Eigenvalues of $\mathcal{H}$ at $\epsilon = 0$:

- $\gamma = -2$ of multiplicity N
- $\gamma = 0$ of multiplicity N
- $\gamma = +1$ of multiplicity ∞

Eigenvalues of $\mathcal{J}\mathcal{H}$ at $\epsilon = 0$:

- $\lambda = 0$ of multiplicity $2N$
- $\lambda = +i$ of multiplicity ∞
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How do zero eigenvalues split?

Stability of solutions in Lyapunov-Schmidt reductions

- First-order splitting of zero eigenvalues of $\mathcal{H}$:

$$\mathcal{M}_1 \mathbf{c} = \gamma \mathbf{c}$$

- First-order splitting of zero eigenvalues of $\mathcal{J}\mathcal{H}$:

$$\mathcal{M}_1 \mathbf{c} = \frac{\lambda^2}{2} \mathbf{c}$$

Stability of solutions in Lyapunov-Schmidt reductions

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- Second-order splitting of zero eigenvalues of $\mathcal{H}$:

$$\mathcal{M}_1 = 0, \qquad \mathcal{M}_2 \mathbf{c} = \gamma \mathbf{c}$$

- Second-order splitting of zero eigenvalues of $\mathcal{J}\mathcal{H}$:

$$\mathcal{M}_1 = 0, \qquad \mathcal{M}_2 \mathbf{c} = \frac{\lambda^2}{2} \mathbf{c} + \lambda \mathcal{L}_2 \mathbf{c}$$

where $M_2^T = M_2$ and $L_2^T = -L_2$.

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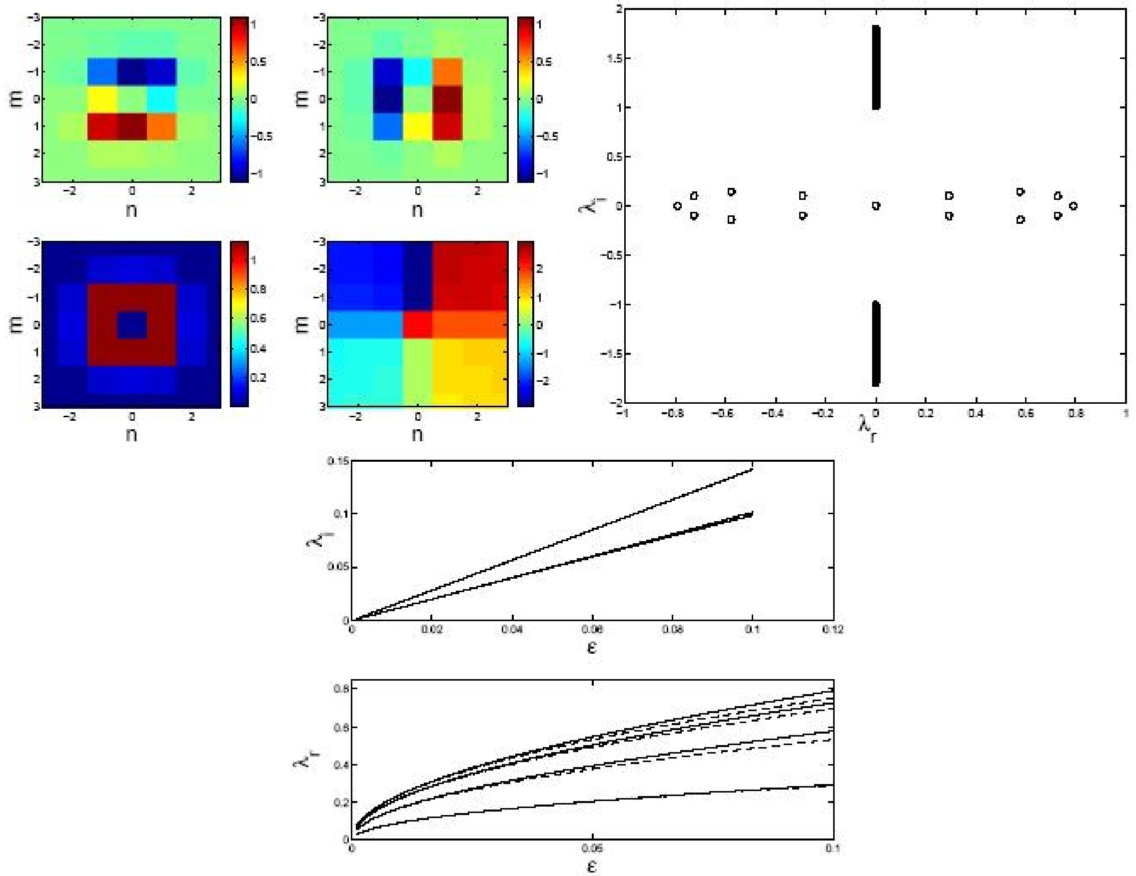
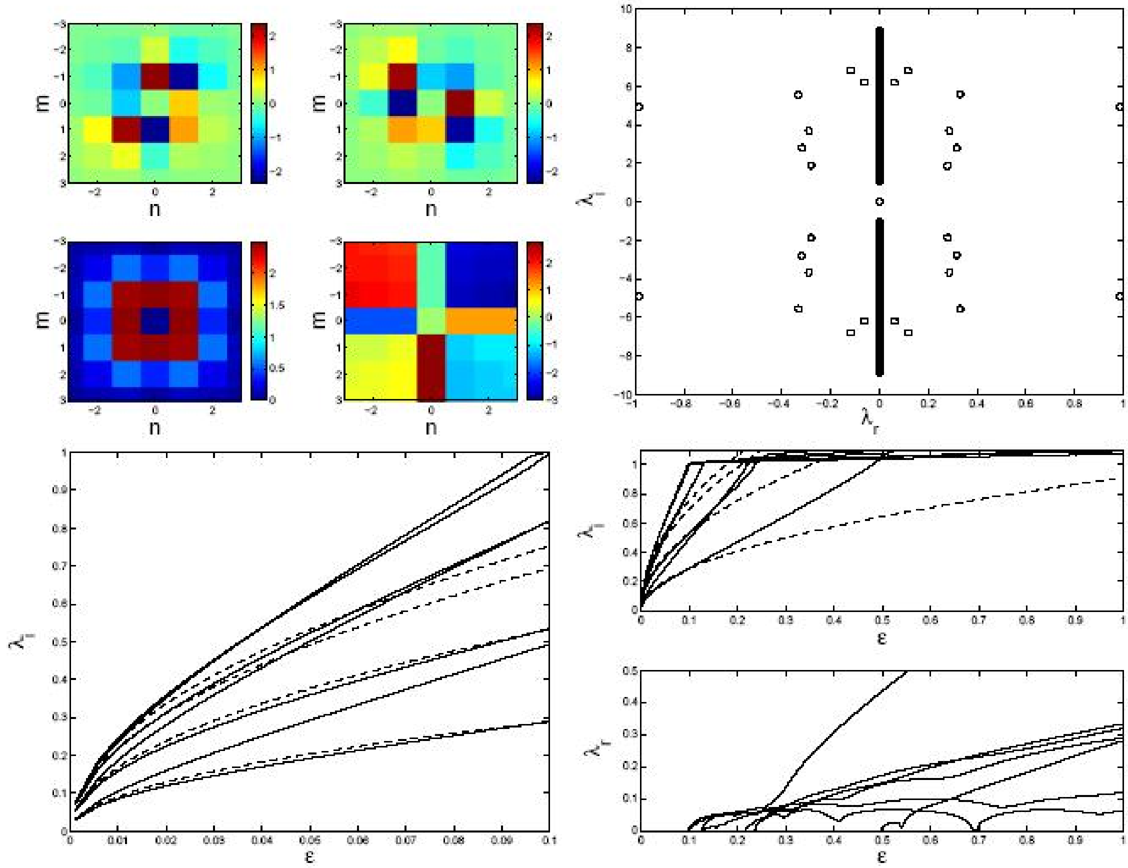
- Six-order splitting : symbolic software algorithm

Numerical analysis: symmetric vortex with $L = 1$ and $M = 2$



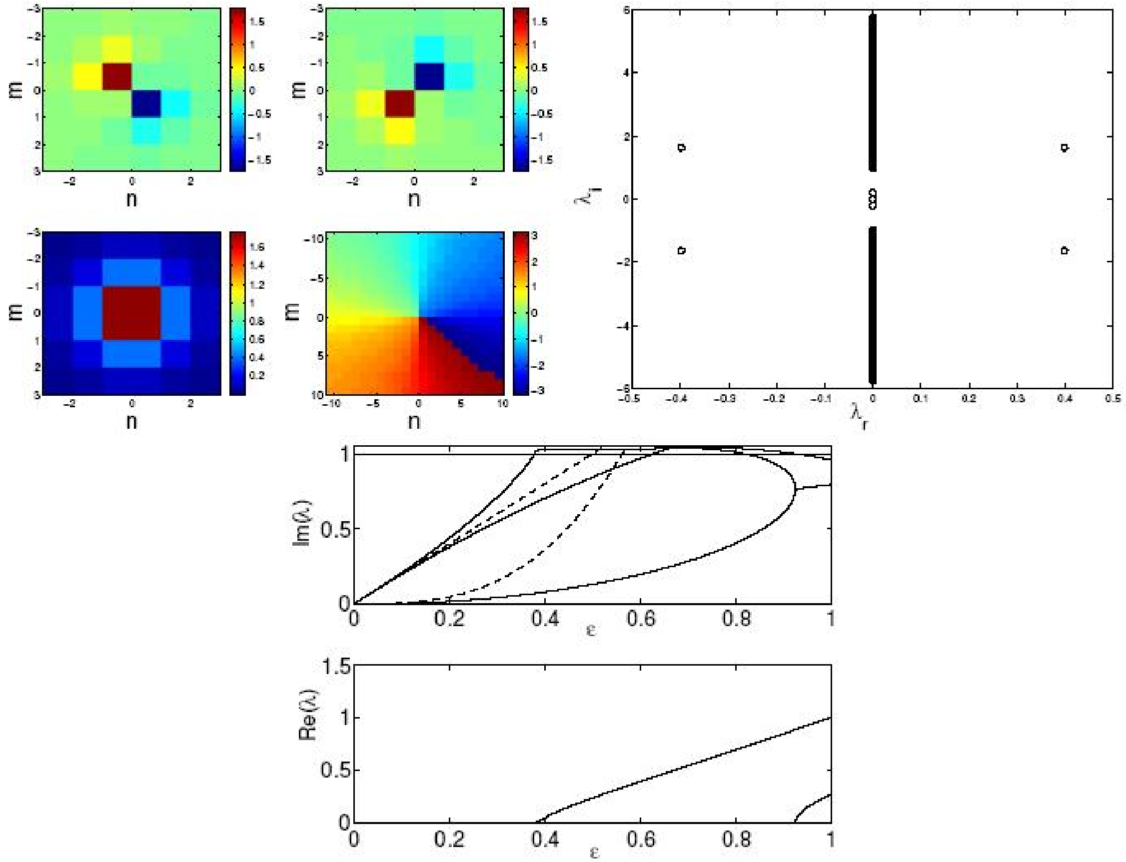
$$\mathcal{M}_1 \mathbf{c} = \gamma \mathbf{c} : \quad n(\mathcal{M}_1) = 0, z(\mathcal{M}_1) = 1, p(\mathcal{M}_1) = 7$$

Numerical analysis: symmetric vortex with $L = 3$ and $M = 2$



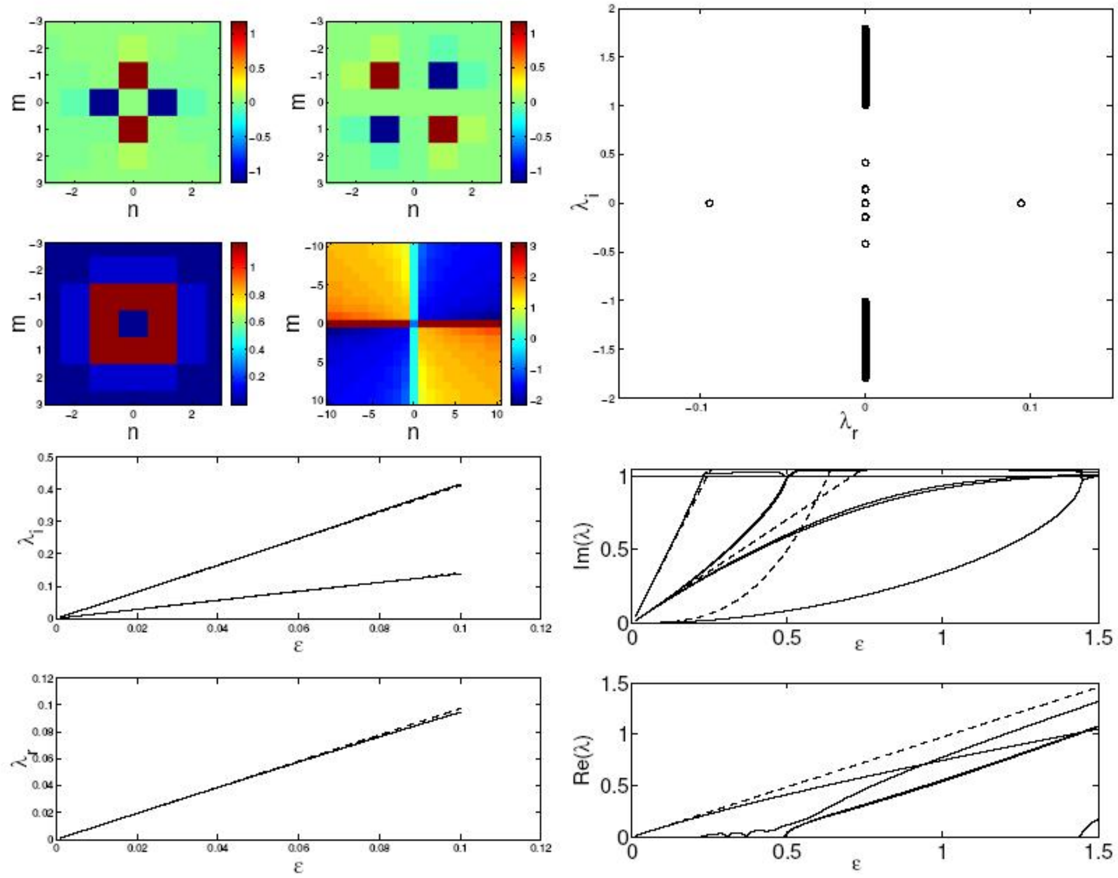
$$\mathcal{M}_1 \mathbf{c} = \gamma \mathbf{c} : \quad n(\mathcal{M}_1) = 7, z(\mathcal{M}_1) = 1, p(\mathcal{M}_1) = 0$$

Numerical analysis: symmetric vortex with $L = M = 1$



$$\mathcal{M}_2 \mathbf{c} = \gamma \mathbf{c} : \quad n(\mathcal{M}_2) = 0, z(\mathcal{M}_2) = 2, p(\mathcal{M}_2) = 2$$

Numerical analysis: symmetric vortex with $L = M = 2$



$$\mathcal{M}_2 \mathbf{c} = \gamma \mathbf{c} : \quad n(\mathcal{M}_2) = 1, z(\mathcal{M}_2) = 2, p(\mathcal{M}_2) = 5$$

Summary:

- Systematic classification of discrete vortices
- Rigorous study of their existence and stability
- Predictions of stable and unstable vortices

contour S_M	vortex of charge L	linearized energy H	stable and unstable eigenvalues
$M = 1$	symmetric $L = 1$	$n(H) = 5$, $p(H) = 2$	$N_r = 0$, $N_i^+ = 1$, $N_i^- = 2$, $N_c = 0$
$M = 2$	symmetric $L = 1$	$n(H) = 8$, $p(H) = 7$	$N_r = 1$, $N_i^+ = 0$, $N_i^- = 0$, $N_c = 3$
$M = 2$	symmetric $L = 2$	$n(H) = 10$, $p(H) = 5$	$N_r = 1$, $N_i^+ = 2$, $N_i^- = 4$, $N_c = 0$
$M = 2$	symmetric $L = 3$	$n(H) = 15$, $p(H) = 0$	$N_r = 0$, $N_i^+ = 0$, $N_i^- = 7$, $N_c = 0$
$M = 2$	asymmetric $L = 1$	$n(H) = 9$, $p(H) = 6$	$N_r = 6$, $N_i^+ = 0$, $N_i^- = 1$, $N_c = 0$
$M = 2$	asymmetric $L = 3$	$n(H) = 14$, $p(H) = 1$	$N_r = 1$, $N_i^+ = 0$, $N_i^- = 6$, $N_c = 0$