Stability of nonlinear waves in integrable Hamiltonian PDEs

Dmitry Pelinovsky
Department of Mathematics, McMaster University, Ontario, Canada
http://dmpeli.math.mcmaster.ca

Workshop "Linear and Nonlinear Dirac Equations"
Como, Italy, February 8-10, 2017

I. Integrable Hamiltonian PDEs

An abstract Hamiltonian PDE can be written in the form

$$\frac{du}{dt} = J\, H'(u), \quad u(t) \in X$$

where $X \subset L^2$ is the phase space, $J^* = -J$ represents the symplectic structure, and $H : X \to \mathbb{R}$ is the Hamilton function.

Example: Korteweg–de Vries (KdV) equation

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \quad u(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

Hamiltonian system in the form

$$\frac{du}{dt} = \frac{\partial}{\partial x}\,\frac{\delta H}{\delta u}, \quad \text{where} \quad H(u) = \frac{1}{2}\int_{\mathbb{R}} \left[\left(\frac{\partial u}{\partial x}\right)^2 - 2u^3\right] dx.$$

I. Integrable Hamiltonian PDEs

An abstract Hamiltonian PDE can be written in the form

$$\frac{du}{dt} = J\, H'(u), \quad u(t) \in X$$

where $X \subset L^2$ is the phase space, $J^* = -J$ represents the symplectic structure, and $H : X \to \mathbb{R}$ is the Hamilton function.

Example: nonlinear Schrödinger (NLS) equation

$$\mathrm{i}\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + 2|u|^2 u = 0, \quad u(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$$

Hamiltonian system in the form

$$\frac{du}{dt} = \mathrm{i}\,\frac{\delta H}{\delta \bar{u}}, \quad \text{where} \quad H(u) = \frac{1}{2}\int_{\mathbb{R}} \left[\left| \frac{\partial u}{\partial x} \right|^2 - 2|u|^4 \right] dx.$$

Class of integrable Hamiltonian PDEs

Korteweg–de Vries (KdV) equation

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \quad u(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

is integrable in the sense of the inverse scattering transform method

- The (smooth) solution $u(t,x)$ is a potential of the Lax operator pair

$$L(u)\psi = \lambda\psi, \quad \frac{\partial \psi}{\partial t} = A(u,\lambda)\psi,$$

  such that λ is (t,x)-independent. The Cauchy problem can be solved by a sequence of direct and inverse scattering transforms.

- Infinitely many conserved quantities exist for smooth solutions.

- Bäcklund–Darboux transformation allows to construct many exact solutions (solitary waves, periodic waves, rogue waves, etc.)

Ablowitz–Kaup–Newell–Segur, Zakharov–Shabat, Fokas, $+\infty$.

New developments for integrable Hamiltonian PDEs

Many classical PDE problems, which were opened in the functional-analytic framework, have been recently solved for the integrable nonlinear PDEs.

Example 1 : Global existence for the derivative NLS equation

$$\begin{cases} iu_t + u_{xx} + i(|u|^2 u)_x = 0, & t > 0, \\ u|_{t=0} = u_0 \in X, \end{cases}$$

where X is some Banach space.

Definition

The Cauchy problem is locally well-posed in X if there exists an unique solution $u(t,\cdot) \in X$ for $t \in (-T, T)$ with finite $T > 0$ and the solution map $u_0 \mapsto u(t,\cdot)$ is continuous. It is globally well-posed if T can be arbitrarily large.

Example 1: Global existence for the DNLS equation

- Tsutsumi & Fukuda (1980) established local well-posedness in $H^s(\mathbb{R})$ with $s > \frac{3}{2}$ and extended solutions globally in $H^2(\mathbb{R})$ for small data in $H^1(\mathbb{R})$
- Hayashi (1993) used gauge transformation of DNLS to a system of semi-linear NLS and established local and global well-posedness in $H^1(\mathbb{R})$ under the constraint $\|u_0\|_{L^2} < \sqrt{2\pi}$.
- Global existence was proved in $H^s(\mathbb{R})$ for $s > \frac{32}{33}$ (Takaoka, 2001), $s > \frac{1}{2}$ (Colliander et al, 2002), and $s = \frac{1}{2}$ (Mio-Wu-Xu, 2011) under the same constraint $\|u_0\|_{L^2} < \sqrt{2\pi}$.

Recent development:
global existence without restriction on the $L^2(\mathbb{R})$ norm.
Liu–Perry–Sulem (2016); P–Shimabukuro (2017).

New developments for integrable nonlinear PDEs

Example 2 : Orbital stability in spaces of low regularity

$$\begin{cases} iu_t + u_{xx} + |u|^2 u = 0, \quad t > 0, \\ u|_{t=0} = u_0 \in X. \end{cases}$$

The Cauchy problem is globally well-posed for $X = L^2(\mathbb{R})$ (Tsutsumi, 1986).

The family of stationary solitary waves

$$u_\omega(t,x) := \sqrt{2\omega}\,\mathrm{sech}\left(\sqrt{\omega}x\right)\ e^{i\omega t},$$

where $\omega > 0$ is arbitrary parameter.

Definition

The solitary wave u_ω is said to be orbitally stable in X if for any $\epsilon > 0$ there is a $\delta > 0$ such that if $\|u(0,\cdot) - u_\omega(0,\cdot)\|_X < \delta$ then

$$\inf_{\theta \in \mathbb{R}} \|u(t,\cdot) - e^{i\theta} u_\omega(t,\cdot)\|_X < \epsilon \quad \text{for} \quad \text{all} \quad t > 0.$$

Example 2 : Orbital stability in spaces of low regularity

- Orbital stability in $H^1(\mathbb{R})$ is proved with the energy method (Lyapunov functions and constrained minimization)
  Weinstein (1985), Shatah–Strauss (1985), Grillakis *et al.* (1987).

- Energy methods do not work in $L^2(\mathbb{R})$ due to lack of control.

- With the Bäcklund–Darboux transformation, orbital and asymptotic stability of solitary waves can be obtained for the NLS equation.
  Mizumachi–P. (2012); Cuccagna–P. (2014); Contreras–P (2014).

New developments for integrable nonlinear PDEs

Example 3 : stability of non-stationary solutions

- N-soliton solutions are orbitally stable in $H^N(\mathbb{R})$
  - KdV [Sachs - Maddocks (1993)]
  - NLS [Kapitula (2006)]
  - Derivative NLS [Le Coz–Wu (2016)]

- Breathers are orbitally stable in $H^2(\mathbb{R})$
  - modified KdV [Alejo–Munoz (2013)]
  - sine-Gordon [Alejo–Munoz (2016)]

In the rest of my talk, I will restrict attention to stability of relative equilibria in Hamiltonian systems (solitary waves, periodic waves) by using energy methods.

New developments for integrable nonlinear PDEs

Example 3 : stability of non-stationary solutions

- N-soliton solutions are orbitally stable in $H^N(\mathbb{R})$
  - KdV [Sachs - Maddocks (1993)]
  - NLS [Kapitula (2006)]
  - Derivative NLS [Le Coz–Wu (2016)]
- Breathers are orbitally stable in $H^2(\mathbb{R})$
  - modified KdV [Alejo–Munoz (2013)]
  - sine-Gordon [Alejo–Munoz (2016)]

In the rest of my talk, I will restrict attention to stability of relative equilibria in Hamiltonian systems (solitary waves, periodic waves) by using energy methods.

II. Stability of relative equilibria in Hamiltonian systems

Consider again an abstract Hamiltonian dynamical system

$$\frac{du}{dt} = J\,H'(u), \quad u(t) \in X$$

where $X \subset L^2$ is the phase space, J is a skew-adjoint operator with a bounded inverse, and $H : X \to \mathbb{R}$ is the Hamilton function.

- Assume existence of the equilibrium $u_0 \in X$ such that $H'(u_0) = 0$.
- Perform linearization $u(t) = u_0 + ve^{\lambda t}$, where λ is the spectral parameter and $v \in X$ satisfies the spectral problem

  $$JH''(u_0)v = \lambda v,$$

  where $H''(u_0) : X \to L^2$ is a self-adjoint Hessian operator.

Main Question

Consider the spectral problem:

$$JH''(u_0)v = \lambda v, \quad v \in X.$$

Question: Is there a relation between unstable eigenvalues of $JH''(u_0)$ and eigenvalues of $H''(u_0)$?

Assumptions of the negative index theory:

- The spectrum of $H''(u_0)$ is strictly positive except for finitely many negative and zero eigenvalues of finite multiplicity.
- The spectrum of $JH''(u_0)$ is purely imaginary except for finitely many unstable eigenvalues.
- Multiplicity of the zero eigenvalue of $JH''(u_0)$ is given by the number of parameters in u_0 (symmetries).

Answer for gradient systems

For a gradient system:

$$\frac{du}{dt} = -F'(u) \quad \Rightarrow \quad \lambda v = -F''(u_0)v,$$

there exists the exact relation between unstable eigenvalues of $-F''(u_0)$ and negative eigenvalues of $F''(u_0)$.

Theorem

The number of unstable eigenvalues of $-F''(u_0)$ is equal to the number of negative eigenvalues of $F''(u_0)$.

What is about Hamiltonian systems?

$$\lambda v = JH''(u_0)v, \quad v \in X.$$

Quadruple Symmetry: If λ is an eigenvalue, so is $-\lambda$, $\bar{\lambda}$, and $-\bar{\lambda}$.

Stability Theorems for Hamiltonian Systems

Consider the spectral stability problem:

$$JH''(u_0)v = \lambda v, \quad v \in X,$$

under the assumptions above on J and $H''(u_0)$.

Orbital Stability Theorem [Grillakis–Shatah–Strauss (1990)]

- Assume no symmetries/zero eigenvalues of $H''(u_0)$. If $H''(u_0)$ has no negative eigenvalues, then $JH''(u_0)$ has no unstable eigenvalues and u_0 is linearly and nonlinearly stable.
- Assume zero eigenvalue of $H''(u_0)$ of multiplicity N and related N symmetries/conserved quantities. If $H''(u_0)$ has no negative eigenvalues under N constraints, then $JH''(u_0)$ has no unstable eigenvalues and u_0 is orbitally stable.

Stability Theorems for Hamiltonian Systems

Consider the spectral stability problem:

$$JH''(u_0)v = \lambda v, \quad v \in X,$$

under the assumptions above on J and $H''(u_0)$.

Orbital Stability Theorem [Grillakis–Shatah–Strauss (1990)]

- Assume no symmetries/zero eigenvalues of $H''(u_0)$. If $H''(u_0)$ has no negative eigenvalues, then $JH''(u_0)$ has no unstable eigenvalues and u_0 is linearly and nonlinearly stable.
- Assume zero eigenvalue of $H''(u_0)$ of multiplicity N and related N symmetries/conserved quantities. If $H''(u_0)$ has no negative eigenvalues under N constraints, then $JH''(u_0)$ has no unstable eigenvalues and u_0 is orbitally stable.

Negative Index Theorem [Kapitula–Kevrekidis–Sandstede (2004)]

Assume no symmetries/zero eigenvalues of $H''(u_0)$. Then,

$$N_{\rm re}(JH''(u_0)) + 2N_{\rm c}(JH''(u_0)) + 2N_{\rm im}^-(JH''(u_0)) = N_{\rm neg}(H''(u_0)) < \infty,$$

where

- $N_{\rm re}$ - number of real unstable eigenvalues;
- $2N_{\rm c}$ - number of complex unstable eigenvalues;
- $2N_{\rm im}^-$ - number neutrally stable eigenvalues of negative Krein signature.

Definition

Suppose that $\lambda \in i\mathbb{R}$ is a simple isolated eigenvalue of $JH''(u_0)$ with the eigenvector v. Then, the sign of the quadratic form

$$\langle H''(u_0)v, v\rangle_{L^2} = \lambda \langle J^{-1}v, v\rangle_{L^2}$$

is called the Krein signature of the eigenvalue λ.

III. Massive Thirring Model (MTM)

The nonlinear Dirac equation (MTM) in the space of one dimension are:

$$\left\{ \begin{array}{l} i(u_t + u_x) + v = 2|v|^2 u, \\ i(v_t - v_x) + u = 2|u|^2 v, \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} i\psi_t - \varphi_x - \psi = (\psi^2 + \varphi^2)\bar{\psi}, \\ i\varphi_t + \psi_x + \varphi = (\psi^2 + \varphi^2)\bar{\varphi}. \end{array} \right.$$

Global solutions exist in $H^1(\mathbb{R})$ [Goodman *et al.* (2003)] or in $L^2(\mathbb{R})$ [Candy (2011), Huh-Moon (2015)].

Three conserved quantities related to symmetries:

$$Q = \int_{\mathbb{R}} \left(|u|^2 + |v|^2\right) dx,$$

$$P = \frac{i}{2} \int_{\mathbb{R}} \left(u\bar{u}_x - u_x\bar{u} + v\bar{v}_x - v_x\bar{v}\right) dx,$$

$$H = \frac{i}{2} \int_{\mathbb{R}} \left(u\bar{u}_x - u_x\bar{u} - v\bar{v}_x + v_x\bar{v}\right) dx + \int_{\mathbb{R}} \left(-v\bar{u} - u\bar{v} + 2|u|^2|v|^2\right) dx,$$

where H is Hamiltonian. The quadratic part of H is sign-indefinite.

III. Massive Thirring Model (MTM)

The nonlinear Dirac equation (MTM) in the space of one dimension are:

$$\left\{ \begin{array}{l} i(u_t + u_x) + v = 2|v|^2 u, \\ i(v_t - v_x) + u = 2|u|^2 v, \end{array} \right. \quad \text{or} \quad \left\{ \begin{array}{l} i\psi_t - \varphi_x - \psi = (\psi^2 + \varphi^2)\bar{\psi}, \\ i\varphi_t + \psi_x + \varphi = (\psi^2 + \varphi^2)\bar{\varphi}. \end{array} \right.$$

Global solutions exist in $H^1(\mathbb{R})$ [Goodman *et al.* (2003)]
or in $L^2(\mathbb{R})$ [Candy (2011), Huh-Moon (2015)].

Three conserved quantities related to symmetries:

$$Q = \int_{\mathbb{R}} \left(|u|^2 + |v|^2\right) dx,$$

$$P = \frac{i}{2} \int_{\mathbb{R}} \left(u\bar{u}_x - u_x\bar{u} + v\bar{v}_x - v_x\bar{v}\right) dx,$$

$$H = \frac{i}{2} \int_{\mathbb{R}} \left(u\bar{u}_x - u_x\bar{u} - v\bar{v}_x + v_x\bar{v}\right) dx + \int_{\mathbb{R}} \left(-v\bar{u} - u\bar{v} + 2|u|^2|v|^2\right) dx,$$

where H is Hamiltonian. The quadratic part of H is sign-indefinite.

Existence of solitary waves

Time-periodic space-localized solutions

$$u(x,t) = U_\omega(x) e^{-i\omega t}, \quad v(x,t) = V_\omega(x) e^{-i\omega t}$$

satisfy a system of stationary Dirac equations. They are known in the closed analytic form

$$\begin{cases} u(x,t) = i \sin(\gamma) \operatorname{sech}\left[x \sin\gamma - i\frac{\gamma}{2}\right] \; e^{-it\cos\gamma}, \\ v(x,t) = -i \sin(\gamma) \operatorname{sech}\left[x \sin\gamma + i\frac{\gamma}{2}\right] \; e^{-it\cos\gamma}. \end{cases}$$

- Translations in x and t can be added as free parameters.
- Constraint $\omega = \cos\gamma \in (-1,1)$ exists because of the gap in the linear spectrum $(-\infty,-1] \cup [1,\infty)$.
- Moving solitons can be obtained from the stationary solitons with the Lorentz transformation.

Orbital stability of Dirac solitons in H^1

The Dirac soliton can not be a constrained minimizer of H.

However, another higher-order Hamiltonian R exists in $H^1(\mathbb{R})$:

$$R = \int_{\mathbb{R}} \Big[|u_x|^2 + |v_x|^2 - \frac{i}{2}(u_x \bar{u} - \bar{u}_x u)(|u|^2 + 2|v|^2) + \dots$$

$$-(u\bar{v} + \bar{u}v)(|u|^2 + |v|^2) + 2|u|^2|v|^2(|u|^2 + |v|^2) \Big] \, dx,$$

in addition to the other conserved quantities H, Q, and P.

Theorem (P–Shimabukuro (2014))

There is $\omega_0 \in (0,1]$ such that for any fixed $\omega = \cos\gamma \in (-\omega_0, \omega_0)$, the Dirac soliton is a local non-degenerate minimizer of R in $H^1(\mathbb{R})$ under the constraints of fixed values of Q and P.

The energy functionals

- Critical points of $H + \omega Q$ for a fixed $\omega \in (-1,1)$ satisfy the stationary MTM equations. After the reduction $(u,v) = (U, \overline{U})$, we obtain the first-order equation

$$i\frac{dU}{dx} - \omega U + \overline{U} = 2|U|^2 U.$$

The MTM soliton $U = U_\omega$ satisfies the first-order equation.

- Critical points of $R + \Omega Q$ for some fixed $\Omega \in \mathbb{R}$ satisfy another system of equations. After the reduction $(u,v) = (U, \overline{U})$, we obtain the second-order equation

$$\frac{d^2 U}{dx^2} + 6i|U|^2 \frac{dU}{dx} - 6|U|^4 U + 3|U|^2 \bar{U} + U^3 = \Omega U.$$

$U = U_\omega$ also satisfies the second-order equation if $\Omega = 1 - \omega^2$.

The energy functionals

- Critical points of $H + \omega Q$ for a fixed $\omega \in (-1, 1)$ satisfy the stationary MTM equations. After the reduction $(u, v) = (U, \overline{U})$, we obtain the first-order equation

$$i \frac{dU}{dx} - \omega U + \overline{U} = 2|U|^2 U.$$

The MTM soliton $U = U_\omega$ satisfies the first-order equation.

- Critical points of $R + \Omega Q$ for some fixed $\Omega \in \mathbb{R}$ satisfy another system of equations. After the reduction $(u, v) = (U, \overline{U})$, we obtain the second-order equation

$$\frac{d^2 U}{dx^2} + 6i|U|^2 \frac{dU}{dx} - 6|U|^4 U + 3|U|^2 \bar{U} + U^3 = \Omega U.$$

$U = U_\omega$ also satisfies the second-order equation if $\Omega = 1 - \omega^2$.

The Lyapunov functional for MTM solitons

We define the conserved energy functional in $H^1(\mathbb{R})$ by

$$\Lambda_\omega := R + (1 - \omega^2) Q, \quad \omega \in (-1, 1),$$

where $Q = \|u\|_{L^2}^2 + \|v\|_{L^2}^2$.

- U_ω is a critical point of Λ_ω.
- The second variation of Λ_ω can be block-diagonalized

$$S^T \Lambda''_\omega(U_\omega) S = \begin{bmatrix} L_+ & 0 \\ 0 & L_- \end{bmatrix},$$

  where L_+ and L_- are 2×2 matrix Schrödinger operators. Chugunova–P (2006); P–Shimabukuro (2014);
- $\Lambda''_\omega(U_\omega)$ has one negative eigenvalue and a double zero eigenvalue for $\omega > 0$ and $\omega < 0$. The zero eigenvalue is quadruple for $\omega = 0$.

Convexity of the energy functional

- Two constraints are added to fix the values of Q and P.
- Two constraints are added to eliminate translation and rotation.
- The Hessian operator $\Lambda''_\omega(U_\omega)$ is strictly positive under the four constraints. The conserved energy functional Λ_ω becomes convex at U_ω in the constrained $H^1(\mathbb{R})$ space.
- The four constraints can be realized by the choice of four modulation parameters in the soliton orbit:

$$\left\{ \begin{array}{l} u(x,t) = i\sin(\gamma)\,\mathrm{sech}\left[x\sin(\gamma) - i\frac{\gamma}{2} - \alpha\right]\; e^{-it\cos(\gamma) - i\beta}, \\ v(x,t) = -i\sin(\gamma)\,\mathrm{sech}\left[x\sin(\gamma) + i\frac{\gamma}{2} - \alpha\right]\; e^{-it\cos(\gamma) - i\beta}, \end{array} \right.$$

with parameters α, β, frequency $\omega := \cos\gamma$, and speed c.

IV. The defocusing nonlinear Schrödinger equation

The cubic NLS equation

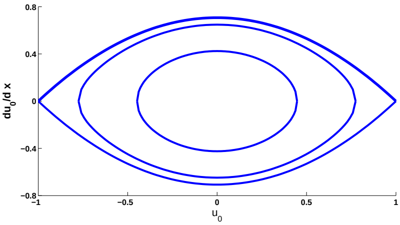
$$i\psi_t + \psi_{xx} - |\psi|^2\psi = 0$$

has long been known for modulational stability of periodic waves.

Periodic waves are of the form $\psi(x,t) = u_0(x)e^{-it}$, where

$$u_0''(x) + (1 - |u_0|^2)u_0 = 0$$

has the exact solution $u_0(x) = \sqrt{1-\mathcal{E}}\,\mathrm{sn}\left(x\frac{\sqrt{1+\mathcal{E}}}{\sqrt{2}}; \sqrt{\frac{1-\mathcal{E}}{1+\mathcal{E}}}\right)$ with $\mathcal{E} \in (0,1)$.



Orbital stability of periodic waves in $H^1_{\rm per}$ or $H^2_{\rm per}$

Periodic waves are constrained minimizers of energy in $H^1_{\rm per}$:

$$E(\psi) = \int \left[|\psi_x|^2 + \frac{1}{2}(1 - |\psi|^2)^2 \right] dx$$

under fixed values of

$$Q(\psi) = \int |\psi|^2 dx, \quad M(\psi) = \frac{i}{2} \int (\bar{\psi}\psi_x - \psi\bar{\psi}_x) dx,$$

if the period of perturbations coincides with the period of waves.
[Gallay–Haragus (2007)]

Periodic waves are also constrained minimizers of the higher-order energy

$$R(\psi) = \int \left[|\psi_{xx}|^2 + 3|\psi|^2|\psi_x|^2 + \frac{1}{2}(\bar{\psi}\psi_x + \psi\bar{\psi}_x)^2 + \frac{1}{2}|\psi|^6 \right] dx,$$

under fixed values of Q and M under the same assumption on the period.

Orbital stability of periodic waves in $H^1_{\rm per}$ or $H^2_{\rm per}$

Periodic waves are constrained minimizers of energy in $H^1_{\rm per}$:

$$E(\psi) = \int \left[|\psi_x|^2 + \frac{1}{2}(1 - |\psi|^2)^2 \right] dx$$

under fixed values of

$$Q(\psi) = \int |\psi|^2 dx, \quad M(\psi) = \frac{i}{2} \int (\bar{\psi}\psi_x - \psi\bar{\psi}_x) dx,$$

if the period of perturbations coincides with the period of waves.
[Gallay–Haragus (2007)]

Periodic waves are also constrained minimizers of the higher-order energy

$$R(\psi) = \int \left[|\psi_{xx}|^2 + 3|\psi|^2|\psi_x|^2 + \frac{1}{2}(\bar{\psi}\psi_x + \psi\bar{\psi}_x)^2 + \frac{1}{2}|\psi|^6 \right] dx,$$

under fixed values of Q and M under the same assumption on the period.

Orbital stability of periodic waves in $H^2_{\rm Nper}$

Periodic waves are not constrained minimizers of neither E nor R if the period of perturbations is multiple to the period of waves.

Nevertheless, there exists a range of values for parameter c such that the energy functional $\Lambda_c := R - cE$ is positively definite at u_0. [Bottman–Deconinck–Nivala (2011)]

Theorem (Gallay–P (2015))

For all $\mathcal{E} \in (0,1)$, the second variation of Λ_c at the periodic wave u_0 is nonnegative for perturbations in $H^2_{\rm Nper}$ only if $c \in [c_-, c_+]$ with

$$c_{\pm} := 2 \pm \frac{2\kappa}{1+\kappa^2}, \quad \kappa = \sqrt{\frac{1-\mathcal{E}}{1+\mathcal{E}}}.$$

Moreover, it is strictly positive up to symmetries in (c_-, c_+) if $\mathcal{E}$ is small.

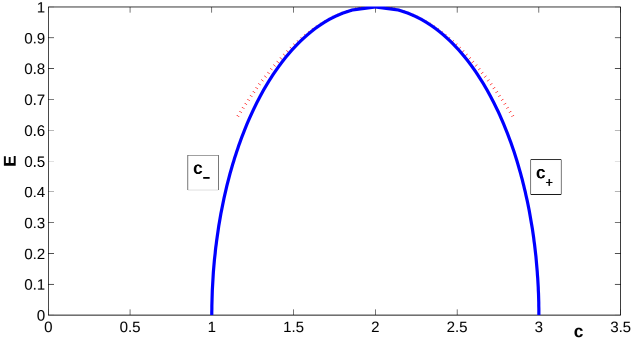


Figure : $(\mathcal{E}, c)$-plane for positivity of the second variation of Λ_c.

A simple perturbative argument

Using the decomposition $\psi = u_0 + u + iv$ with real-valued perturbation functions u and v, we can write

$$\Lambda_c(\psi) - \Lambda_c(u_0) = \langle K_+(c)u, u\rangle_{L^2} + \langle K_-(c)v, v\rangle_{L^2} + \text{cubic terms}$$

where

$$K_+(c)\partial_x u_0 = 0 \quad \text{and} \quad K_-(c)u_0 = 0.$$

If $u_0 = 0$ (periodic wave of zero amplitude), then

$$\begin{aligned} \langle K_\pm(c)u, u\rangle_{L^2} &= \int_{\mathbb{R}} \left[u_{xx}^2 - cu_x^2 + (c-1)u^2\right] dx \\ &= \int \left(u_{xx} + \frac{c}{2}u\right)^2 dx - \left(1 - \frac{c}{2}\right)^2 \int u^2 dx. \end{aligned}$$

Then, $\langle K_\pm(c)u, u\rangle_{L^2} \geq 0$ if $c = 2$. By perturbative computations, one can find (c_-, c_+) near $c = 2$ for $\mathcal{E} < 1$.

Orbital stability of periodic waves in $H^2_{\rm Nper}$

Theorem (Gallay–P (2015))

Assume that $\psi_0 \in H^2_{\rm Nper}$ and consider the global-in-time solution ψ to the cubic NLS equation with initial data ψ_0. For any $\epsilon > 0$, there is $\delta > 0$ s.t. if

$$\|\psi_0 - u_0\|_{H^2_{\rm Nper}} \leq \delta,$$

then, for any $t \in \mathbb{R}$, there exist numbers $\xi(t)$ and $\theta(t)$ such that

$$\|e^{i(t+\theta(t))}\psi(\cdot + \xi(t), t) - u_0\|_{H^2_{\rm Nper}} \leq \epsilon.$$

Moreover, ξ, θ are continuous and $|\dot{\xi}(t)| + |\dot{\theta}(t)| \leq C\epsilon$.

V. The Kadomtsev–Petviashvili (KP) equation

The 2D generalization of the KdV equation is the KP equation:

$$(u_t + 6uu_x + u_{xxx})_x = \pm u_{yy},$$

where the plus/minus sign corresponds to KP-I/KP-II equations.

Periodic waves $u = v(x + ct)$ of the cnoidal form satisfies the 1D KdV equation. Transverse stability is determined for small 2D perturbations w:

$$(w_t + cw_x + 6(vw)_x + w_{xxx})_x = \pm w_{yy}.$$

KP-I: Periodic and solitary waves are transversely unstable [Johnson–Zumbrun (2010); Rousset–Tzvetkov (2011); Hakkaev (2012)]

KP-II: Solitary waves are transversely stable [Mizumachi–Tzvetkov (2012); Mizumachi (2015) (2016)]

KP-II: Stability of periodic waves is open [Haragus (2010)].

Conserved quantities for KP-II equation

The momentum of KP-II equation is

$$Q(u) = \int u^2 dxdy$$

The energy of KP-II equation is sign-indefinite near zero:

$$E(u) = \int \left[u_x^2 - 2u^3 - (\partial_x^{-1} u_y)^2 \right] dxdy.$$

The higher-order energy is still sign-indefinite near zero:

$$R(u) = \int \left[u_{xx}^2 - 10uu_x^2 + 5u^4 - \frac{10}{3} u_y^2 + \frac{5}{9} (\partial_x^{-2} u_{yy})^2 + \frac{10}{3} u^2 \partial_x^{-2} u_{yy} + \ldots \right] dxdy.$$

Molinet–Saut–Tzvetkov (2007)

The previous approach to characterization periodic waves as constrained energy minimizers for a linear combination of $E(u)$ and $R(u)$ fails.

Commuting operators via symplectic operators

1D periodic waves $u(t,x) = v(x+ct)$ are critical points of $E(u) + cQ(u)$ with the Hessian operator

$$L_{c,p} = -\partial_x^2 - c - 6v(x) + p^2 \partial_x^{-2},$$

where p is the transverse wave number for the 2D perturbation $w(x,y) = W(x)e^{ipy}$.

Search for the commuting self-adjoint operator $M_{c,p}$ in

$$L_{c,p} \partial_x M_{c,p} = M_{c,p} \partial_x L_{c,p},$$

where ∂_x defines the symplectic operator for the KP-II equation.

Theorem (Haragus–Li-P (2017))

Assume that $M_{c,p} \geq 0$ and the kernel of $M_{c,p}$ is contained in the kernel of $L_{c,p}$. The spectrum of $\partial_x L_{c,p}$ is purely imaginary.

Algorithmic search of the commuting operator

We are looking for an operator $M_{c,p}$ to satisfy the commutability relation

$$L_{c,p}\partial_x M_{c,p} = M_{c,p}\partial_x L_{c,p}.$$

Since 1D periodic waves $u = v(x + ct)$ are also critical points of $R(u)$, the Hessian operator $M_{c,p}$ related to $R(u)$ satisfies this commutability relation. The operator $M_{c,p}$ is given by

$$\begin{aligned} M_{c,p} &= \partial_x^4 + 10\partial_x v(x)\partial_x - 10cv(x) - c^2 \\ &\quad -\frac{10}{3}p^2\left(1 + v(x)\partial_x^{-2} + \partial_x^{-1}v(x)\partial_x^{-1} + \partial_x^{-2}v(x)\right) + \frac{5}{9}p^4\partial_x^{-4}. \end{aligned}$$

Lemma

For every $p \neq 0$, no value of $b \in \mathbb{R}$ exists such that $M_{c,p} - bL_{c,p}$ is positive. Moreover, the number of negative eigenvalues quickly grows in $L^2_{\rm Nper}$ with larger N.

Algorithmic search of the commuting operator

We are looking for an operator $M_{c,p}$ to satisfy the commutability relation

$$L_{c,p}\partial_x M_{c,p} = M_{c,p}\partial_x L_{c,p}.$$

By using symbolic computations, we have found another choice of the commuting operator

$$M_{c,p} = \partial_x^4 + 10\partial_x v(x)\partial_x - 10cv(x) - c^2 + \frac{5}{3}p^2\left(1 + c\partial_x^{-2}\right).$$

Lemma

The operator $M_{c,p} + 2cL_{c,p}$ is positive in $L^2_{\rm Nper}$ for every $p \in \mathbb{R}$ and $N \in \mathbb{N}$.

The periodic travelling wave v of the KP-II equation is spectrally stable with respect to two-dimensional bounded perturbations.

Conclusion

- Spectral stability theory is well-developed for relative equilibria in Hamiltonian systems, when the Hessian operators have finitely many negative eigenvalues.
- Orbital stability holds in Hamiltonian systems if the relative equilibrium is a non-degenerate minimum of energy under constraints of fixed mass and momentum.
- For many integrable PDEs (MTM, NLS, KdV), one can use higher-order Hamiltonians to conclude on orbital stability of nonlinear waves.
- For the KP-II equation (in 2D), one can find positive-definite operator unrelated to conserved quantities in order to conclude on spectral stability of nonlinear waves.

The END.