Stability of periodic waves in the defocusing cubic NLS equation

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CMS Winter Meeting, December 5-8, 2014
Hamilton, ON, Canada

Higher-order conserved quantities associated with nonlinear integrable equations have been used in analysis of orbital stability of nonlinear waves:

- Orbital stability of n-solitons in $H^n(\mathbb{R})$ was proved by Sachs - Maddocks (1993) for KdV and by Kapitula (2006) for NLS.
- Orbital stability of breathers in $H^2(\mathbb{R})$ was proved by Alejo–Munoz (2013) for the modified KdV equation.
- Orbital stability of Dirac solitons in $H^1(\mathbb{R})$ was proved by P–Shimabukuro (2014) for the massive Thirring model.
- Orbital stability of periodic waves with respect to subharmonic perturbations was considered by Deconinck and collaborators for KdV (2010) and defocusing NLS (2011).

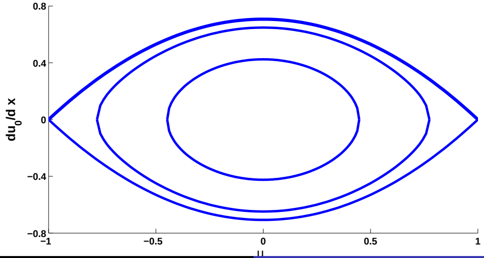
The defocusing NLS equation

$$i\psi_t + \psi_{xx} - |\psi|^2\psi = 0, \quad \psi = \psi(x,t).$$

Periodic waves exist in the form $\psi(x,t) = u_0(x)e^{-it}$, where

$$\frac{d^2 u_0}{dx^2} + (1 - |u_0|^2)u_0 = 0,$$

in fact, in the explicit form $u_0(x) = \sqrt{1-\mathcal{E}}\,\mathrm{sn}\left(x\frac{\sqrt{1+\mathcal{E}}}{\sqrt{2}}; \sqrt{\frac{1-\mathcal{E}}{1+\mathcal{E}}}\right)$,
where $\mathcal{E} \in (0,1)$ is a free parameter.



Periodic waves are critical points of the energy

$$E(\psi) = \int \left[|\psi_x|^2 + \frac{1}{2}(1 - |\psi|^2)^2 \right] dx$$

However, they are not minimizers of the energy even in the space of periodic perturbations.

- Gallay–Haragus (2007) proved orbital stability w.r.t. perturbations with the same period as $|u_0|$ by using constraints from lower-order conserved quantities:

$$Q(\psi) = \int |\psi|^2 dx, \quad M(\psi) = \frac{i}{2} \int (\bar{\psi}\psi_x - \psi\bar{\psi}_x) dx.$$

The periodic waves are constrained minimizers of the energy.

- Bottman–Deconinck–Nivala (2011) proved spectral stability of periodic waves in the Floquet–Bloch theory and used the higher-order conserved quantity

$$R(\psi) = \int \left[|\psi_{xx}|^2 + 3|\psi|^2|\psi_x|^2 + \frac{1}{2}(\bar{\psi}\psi_x + \psi\bar{\psi}_x)^2 + \frac{1}{2}|\psi|^6 \right] dx,$$

  to show that the periodic waves are critical points of the higher-order energy $S := R - \frac{1}{2}(3 - \mathcal{E}^2)Q$.

- Periodic waves are not minimizers of neither E nor S. Nevertheless, the energy functional $\Lambda_c := S - cE$ is claimed to be positively definite at u_0 for some values of parameter c.

- Motivation for our work is to understand the constraints on c and to prove the claim by using rigorous PDE analysis.

Lemma

There exists $\mathcal{E}_0 \in (0,1)$ *s.t. for all* $\mathcal{E} \in (\mathcal{E}_0, 1)$*, there exist values* c_- *and* c_+ *in the range* $1 < c_- < 2 < c_+ < 3$ *s.t. the second variation of* Λ_c *at the periodic wave* u_0 *is nonnegative for perturbations in* $H^2(\mathbb{R})$ *if* $c \in (c_-, c_+)$*. Moreover,*

$$c_\pm = 2 \pm \sqrt{2(1-\mathcal{E})} + \mathcal{O}(1-\mathcal{E}) \quad \text{as} \quad \mathcal{E} \to 1.$$

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$$c_\pm = 2 \pm \sqrt{2(1-\mathcal{E})} + \mathcal{O}(1-\mathcal{E}) \quad \text{as} \quad \mathcal{E} \to 1.$$

Lemma

For all $\mathcal{E} \in (0,1)$, the second variation of Λ_c at the periodic wave u_0 is nonnegative for perturbations in $H^2(\mathbb{R})$ only if $c \in [c_-, c_+]$ with

$$c_\pm := 2 \pm \frac{2\kappa}{1+\kappa^2}, \quad \kappa = \sqrt{\frac{1-\mathcal{E}}{1+\mathcal{E}}}.$$

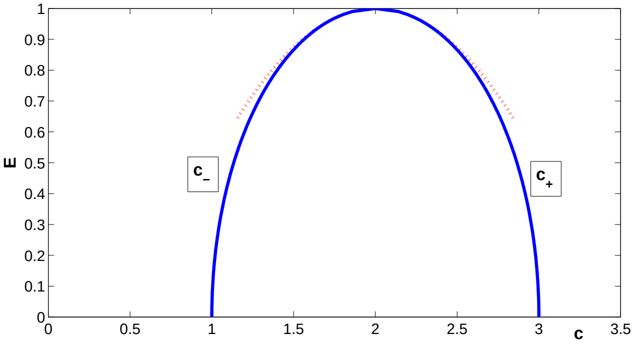

Figure : $(\mathcal{E}, c)$-plane for positivity of the second variation of Λ_c.

Using the decomposition $\psi = u_0 + u + iv$ with real-valued perturbation functions u and v, we can write

$$\Lambda_c(\psi) - \Lambda_c(u_0) = \langle K_+(c)u, u\rangle_{L^2} + \langle K_-(c)v, v\rangle_{L^2} + \text{cubic terms}$$

where

$$K_+(c)\partial_x u_0 = 0 \quad \text{and} \quad K_-(c)u_0 = 0.$$

Lemma

Fix $c = 2$. For any $\mathcal{E} \in (0,1)$ and any $v \in H^2(\mathbb{R})$, we have

$$\langle K_-(2)v, v\rangle_{L^2} = \|v_{xx} + (1 - u_0^2)v\|_{L^2}^2 + \|u_0 v_x - u_0' v\|_{L^2}^2.$$

Moreover, $\langle K_-(2)v, v\rangle_{L^2} = 0$ if and only if $v = Cu_0$.

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Theorem

Assume that $\psi_0 \in H^2_{\rm per}(0,T)$, where T is a multiple of the period of u_0, and consider the global-in-time solution ψ to the cubic NLS equation with initial data ψ_0. For any $\epsilon > 0$, there is $\delta > 0$ s.t. if

$$\|\psi_0 - u_0\|_{H^2_{\rm per}(0,T)} \leq \delta,$$

then, for any $t \in \mathbb{R}$, there exist numbers $\xi(t)$ and $\theta(t)$ such that

$$\|e^{i(t+\theta(t))}\psi(\cdot + \xi(t), t) - u_0\|_{H^2_{\rm per}(0,T)} \leq \epsilon.$$

Let us give a simple argument why $\Lambda_c = S - cE$ can be positive definite at the periodic wave u_0. For $\mathcal{E} = 1$, we have $u_0 = 0$, and

$$\langle K_\pm(c)u, u\rangle_{L^2} = \int_{\mathbb{R}} \left[u_{xx}^2 - cu_x^2 + (c-1)u^2\right] dx.$$

Integration by parts yields

$$\langle K_\pm(c)u, u\rangle_{L^2} = \int \left(u_{xx} + \frac{c}{2}u\right)^2 dx - \left(1 - \frac{c}{2}\right)^2 \int u^2 dx,$$

which is non-negative if $c = 2$.

However, the case $\mathcal{E} = 1$ is degenerate, hence perturbation arguments are needed to unfold the degeneracy for $\mathcal{E} < 1$.

To run perturbation arguments, we normalize the period:

$$u_0(x) = U(z), \quad z = \ell x, \quad U(z+2\pi) = U(z).$$

The function $U(z)$ satisfies

$$\ell^2 \frac{d^2 U}{dz^2} + U - U^3 = 0 \quad \Rightarrow \quad \ell^2 \left(\frac{dU}{dz} \right)^2 = \frac{1}{2} \left[(1-U^2)^2 - \mathcal{E}^2 \right].$$

Lemma

The map $(0,1) \ni \mathcal{E} \mapsto (\ell, U) \in \mathbb{R} \times H^2_{\rm per}(0,2\pi)$ can be uniquely parameterized by the small parameter a as $\mathcal{E} \to 1$ such that

$$\mathcal{E} = 1 - a^2 + \mathcal{O}(a^4), \quad \ell^2 = 1 - \frac{3}{4} a^2 + \mathcal{O}(a^4), \quad U(z) = a\cos(z) + \mathcal{O}(a^3).$$

Operators $K_{\pm}(c)$ from $H^4(\mathbb{R})$ to $L^2(\mathbb{R})$ have 2π-periodic coefficients in variable z. By Floquet–Bloch theory, λ belongs to the purely continuous spectrum of $K_{\pm}(c)$ in $L^2(\mathbb{R})$ if there exists a bounded solution of the spectral problem

$$K_{\pm}(c)u(z,k) = \lambda u(z,k), \quad u(z,k) := e^{ikz} w(z,k),$$

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If $a = 0$, then the Floquet–Bloch spectrum of

$$K_\pm(c) = \partial_z^4 + c\partial_z^2 + (c-1)$$

is found from the Fourier series:

$$\lambda_n^0(k) = (n+k)^4 - c(n+k)^2 + c - 1, \quad n \in \mathbb{Z}.$$

If $c = 2$, then $\lambda_n^0(k) = 0$ if and only if $k = 0$ and $n = \pm 1$.

Lemma

If $a > 0$ is sufficiently small and $c \in (c_-, c_+)$, where $c_\pm = 2 \pm \sqrt{2}a + \mathcal{O}(a^2)$, the operator $K_\pm(c)$ has exactly one Floquet–Bloch band denoted by $\mathrm{range}(\lambda^\pm_{-1}(k))$ that touches the origin at $k = 0$, while all other bands are strictly positive.

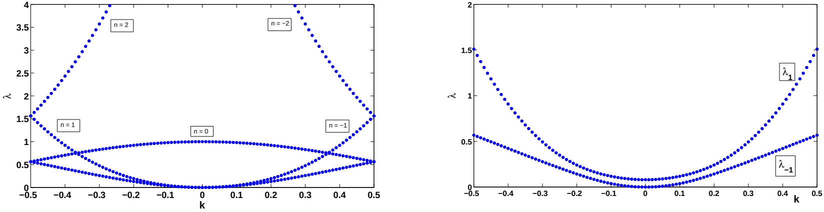

Figure : Spectral bands for $c = 2$ and $a = 0$ (left). Degenerate spectral bands for $c = 2$ and $a = 0.2$ (right).

Consider the operator $K_-(c)$ and the spectral band $\lambda_{-1}^-(k)$ that touches zero at $k = 0$ because $K_-(c)u_0 = 0$.

Lemma

Fix $\mathcal{E} \in (0, 1)$ and assume that u_0 is the only 2π-periodic solution of $K_-(c)w = 0$. Then, λ_{-1}^- is C^2 near $k = 0$ with $\lambda_{-1}^-(0) = \lambda_{-1}^{-\prime}(0) = 0$, and

$$\lambda_{-1}^{-\prime\prime}(0) = \frac{2\ell^2\kappa^2 K(\kappa)(4\kappa^2 - (c-2)^2(1+\kappa^2)^2)}{(1+\kappa^2)(K(\kappa)-E(\kappa))\left(2\kappa^2 + (c-2)(1+\kappa^2)\left(1 - \frac{E(\kappa)}{K(\kappa)}\right)\right)}.$$

Corollary

$\lambda_{-1}^{-\prime\prime}(0) > 0$ if $c \in (c_-, c_+)$, where $c_\pm$ are defined by

$$c_\pm := 2 \pm \frac{2\kappa}{1+\kappa^2}, \quad \kappa = \sqrt{\frac{1-\mathcal{E}}{1+\mathcal{E}}}.$$

The perturbative arguments only apply to the periodic waves of small amplitudes (when $\mathcal{E}$ is close to 1). The question is how to continue these arguments to large-amplitude waves.

Lemma

Fix $c = 2$. For any $\mathcal{E} \in [0,1]$ and any $v \in H^2(\mathbb{R})$, we have

$$\langle K_-(2)v, v\rangle_{L^2} = \|L_- v\|_{L^2}^2 + \|u_0 v_x - u_0' v\|_{L^2}^2,$$

where $L_- := -\partial_x^2 + u_0^2(x) - 1$. Note that $L_- u_0 = K_-(c)u_0 = 0$.

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where $L_- := -\partial_x^2 + u_0^2(x) - 1$. Note that $L_- u_0 = K_-(c)u_0 = 0$.

When we apply the same idea to $K_+(c)$ with $c = 2$, we only obtain

$$\langle K_+(2)u, u\rangle_{L^2} = \|L_+ u\|^2_{L^2} - \int \left[u_0^2 u_x^2 - 3u_0^2 u^2 + 5u_0^4 u^2\right] dx,$$

where $L_+ := -\partial_x^2 + 3u_0^2(x) - 1$. Note that $L_+ u_0' = K_+(c)u_0' = 0$.

In order to obtain the positivity of $K_+(2)$, we apply the following chain of arguments.

1. Operators $L_\pm$ and $K_\pm(c)$ compute for any $c \in \mathbb{R}$ as follows:

$$L_- K_+(c) = K_-(c) L_+, \quad L_+ K_-(c) = K_+(c) L_-.$$

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   This follows from commutability of the evolution flows of the integrable NLS hierarchy.

2. Bounded solutions of $L_\pm$ are given as follows:
   - If $u \in L^\infty(\mathbb{R}) \cap H^2_{\rm loc}(\mathbb{R})$ satisfies $L_+ u = 0$, then $u = C\partial_x u_0$.
   - If $v \in L^\infty(\mathbb{R}) \cap H^2_{\rm loc}(\mathbb{R})$ satisfies $L_- v = 0$, then $v = C u_0$.

   This follows from analysis of Schrödinger operators $L_\pm$ with periodic coefficients.

3. If $v \in L^\infty(\mathbb{R}) \cap H^4_{\rm loc}(\mathbb{R})$ satisfies $K_-(2)v = 0$, then $v = Cu_0$.

If $K_-(2)v = 0$, then for every $N \in \mathbb{N}$ including $N \to \infty$,

$$\begin{aligned}
0 &= \frac{1}{N} \int_0^{NT_0} \left(|L_- v|^2 + |u_0 v_x - u_0' v|^2 \right) dx + \frac{1}{N} \mathrm{b.v.}|_{x=0}^{x=NT_0} \\
&= \int_0^{T_0} \left(|L_- v|^2 + |u_0 v_x - u_0' v|^2 \right) dx + \frac{1}{N} \mathrm{b.v.}|_{x=0}^{x=NT_0}.
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3 If $v \in L^\infty(\mathbb{R}) \cap H^4_{\rm loc}(\mathbb{R})$ satisfies $K_-(2)v = 0$, then $v = Cu_0$.

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4 If $u \in L^\infty(\mathbb{R}) \cap H^4_{\rm loc}(\mathbb{R})$ solves $K_+(2)u = 0$, then $u = C\partial_x u_0$.

Indeed, $K_-(2)L_+ u = L_- K_+(2)u = 0$ implies $L_+ u = Bu_0$. Then, $u = BU + C\partial_x u_0$, where $U = L_+^{-1} u_0$ exists. However, $K_+(2)u = BK_+(2)U$ with $K_+(2)U \neq 0$, hence $B = 0$.

5 For every $\mathcal{E} \in (0,1)$, all spectral bands of the operator $K_+(2)$ cannot touch $\lambda = 0$ except for the lowest band, which touches $\lambda = 0$ exactly at $k = 0$, with $u = C\partial_x u_0$.

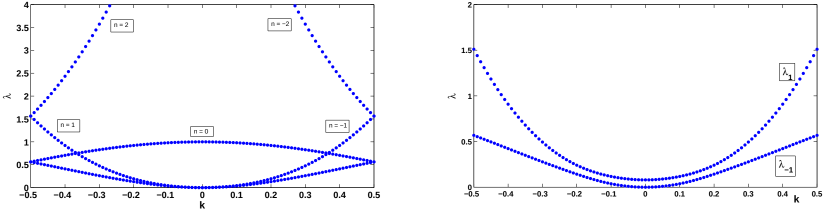

Figure : Spectral bands for $c = 2$ and $a = 0$ (left). Degenerate spectral bands for $c = 2$ and $a = 0.2$ (right).

Let T be a multiple of the period T_0 of the periodic wave u_0. For the unique global solution $\psi(x,t)$ of the cubic defocusing NLS equation in $H^2_{\rm per}(0,T)$, we write

$$e^{i\theta(t)+it}\psi(x+\xi(t),t) = u_0(x) + u(x,t) + iv(x,t),$$

where real u and v satisfy the orthogonality conditions

$$\langle \partial_x u_0, u(\cdot,t)\rangle_{L^2_{\rm per}} = 0, \quad \langle u_0, v(\cdot,t)\rangle_{L^2_{\rm per}} = 0, \quad t \in \mathbb{R}.$$

The functions $\theta(t)$ and $\xi(t)$ are uniquely determined by the orthogonality conditions if $\|u\|_{H^2_{\rm per}}$ and $\|v\|_{H^2_{\rm per}}$ are small. From coercivity of the quadratic forms, we obtain

$$\langle K_+(2)u, u\rangle_{L^2_{\rm per}} \geq C_+\|u\|^2_{H^2_{\rm per}}, \quad \langle K_-(2)v, v\rangle_{L^2_{\rm per}} \geq C_-\|v\|^2_{H^2_{\rm per}}.$$

The rest follows from the energy conservation

$$\begin{aligned}
\Delta\Lambda_{c=2} &:= \Lambda_{c=2}(\psi(\cdot,t)) - \Lambda_{c=2}(u_0) \\
&= \Lambda_{c=2}(e^{i\theta(t)+it}\psi(\cdot+\xi(t),t)) - \Lambda_{c=2}(u_0) \\
&= \langle K_+(2)u,u\rangle_{L^2_{\rm per}} + \langle K_-(2)v,v\rangle_{L^2_{\rm per}} + N(u,v).
\end{aligned}$$

If $\|\psi_0 - u_0\|_{H^2_{\rm per}(0,T)} \leq \delta$, then $|\Delta\Lambda_{c=2}| \leq C\delta^2$. From coercivity of the quadratic part and smallness of the cubic part $N(u,v)$, we obtain for all $t \in \mathbb{R}$:

$$\|e^{i\theta(t)+it}\psi(\cdot+\xi(t),t) - u_0\|_{H^2_{\rm per}} \leq C|\Delta\Lambda_{c=2}|^{1/2} \leq C\delta =: \epsilon.$$

Open questions:

- Orbital stability of periodic waves with respect to perturbations in $H^2(\mathbb{R})$.
- Extension of this analysis for non-real periodic waves with a nontrivial phase.
- Extension of this analysis to other integrable equations (e.g. the Korteweg–de Vries equation).

Reference:

- Th. Gallay and D.P., J. Diff. Eqs. (2014), accepted.