Edge-Localized States on Metric Graphs in the limit of large mass

Dmitry Pelinovsky,
Department of Mathematics, McMaster University
http://dmpeli.math.mcmaster.ca

Collaborators:
Gregory Berkolaiko, Texas A & M University, USA
Jeremy Marzuola, University of North Carolina, USA

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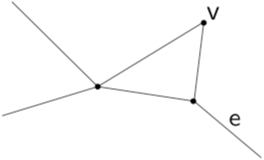
Nonlinear Schrödinger equation on metric graphs



A metric graph $\Gamma = \{E, V\}$ is given by a set of edges E and vertices V, with a metric structure on each edge.

Nonlinear Schrödinger equation on a graph Γ:

$$i\Psi_t = -\Delta\Psi - 2|\Psi|^2\Psi, \quad x \in \Gamma,$$

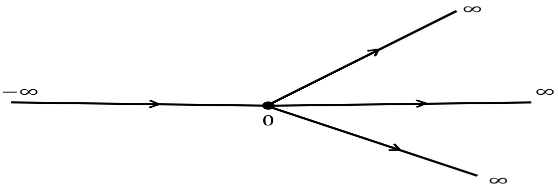
where Δ is the graph Laplacian and $\Psi(t,x)$ is defined componentwise on edges subject to Neumann–Kirchhoff boundary conditions at vertices:

$$\begin{cases} \Psi(v) \text{ is continuous} & \text{for every } v \in V, \\ \sum_{e\sim v} \partial\Psi_e(v) = 0, & \text{for every } v \in V, \end{cases}$$

where $e \sim v$ denotes all edges $e \in E$ adjacent to $v \in V$.

Example: a star graph

A star graph is the union of N half-lines connected at a single vertex. For $N = 2$, the graph is the line $\mathbb{R}$. For $N = 3$, the graph is a Y-junction.



Function spaces are defined componentwise:

$$L^2(\Gamma) = L^2(\mathbb{R}^-) \oplus \underbrace{L^2(\mathbb{R}^+) \oplus \cdots \oplus L^2(\mathbb{R}^+)}_{\text{(N-1) elements}},$$

subject to the Neumann–Kirchhoff conditions at a single vertex:

$$H^1_\Gamma := \{\Psi \in H^1(\Gamma) : \quad \psi_1(0) = \psi_2(0) = \cdots = \psi_N(0)\}$$

$$H^2_\Gamma := \{\Psi \in H^2(\Gamma) \cap H^1_\Gamma : \quad \psi_1'(0) = \sum_{j=2}^{N} \psi_j'(0)\},$$

NLS on the metric graph Γ

The Cauchy problem for the NLS flow:

$$\begin{cases} i\Psi_t = -\Delta\Psi - 2|\Psi|^2\Psi, \\ \Psi|_{t=0} = \Psi_0. \end{cases}$$

Lemma. The Cauchy problem is locally and globally well-posed for $\Psi_0 \in H^1_\Gamma$. Moreover, the mass

$$Q(\Psi) = \|\Psi\|^2_{L^2(\Gamma)}$$

and the energy

$$E(\Psi) = \|\Psi'\|^2_{L^2(\Gamma)} - \|\Psi\|^4_{L^4(\Gamma)},$$

are constants in time for $\Psi \in C(\mathbb{R}, H^1_\Gamma)$.

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$E(\Psi)$ is coercive in $H^1(\Gamma)$ thanks to Gagliardo–Nirenberg inequality:

$$\|\Psi\|^4_{L^4(\Gamma)} \leq C_\Gamma \|\Psi'\|_{L^2(\Gamma)} \|\Psi\|^3_{L^2(\Gamma)},$$

where $C_\Gamma > 0$ depends on Γ only.

Ground state

Ground state is a standing wave of smallest energy E at fixed mass Q,

$$\mathcal{E}_q = \inf\{E(u) : \quad u \in H^1_\Gamma, \quad Q(u) = q\}.$$

Euler–Lagrange equation for the standing waves:

$$-\Delta\Phi - 2|\Phi|^2\Phi = \Lambda\Phi,$$

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Infimum $\mathcal{E}_q$ exists for every $q > 0$ thanks to Gagliardo–Nirenberg inequality.

Theorem. (Adami–Serra–Tilli, 2015) If Γ is unbounded and contains at least one half-line, then

$$\min_{\phi \in H^1(\mathbb{R}^+)} E(u;\mathbb{R}^+) \leq \mathcal{E}_q \leq \min_{\phi \in H^1(\mathbb{R})} E(u;\mathbb{R})$$

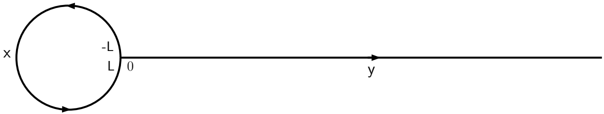
Infimum may not be attained by any of the standing waves Φ if the graph Γ is unbounded.

Ground state on the unbounded graphs

Theorem. (Adami–Serra–Tilli, 2016) If Γ consists of only one half-line, then

$$\mathcal{E}_q < \min_{\phi \in H^1(\mathbb{R})} E(u; \mathbb{R})$$

and the infimum is attained.



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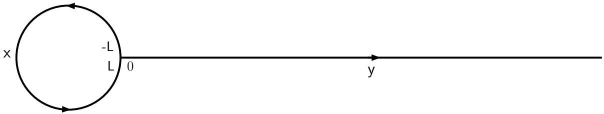


If Γ consists of more than two half-lines and is *connective to infinity*, then

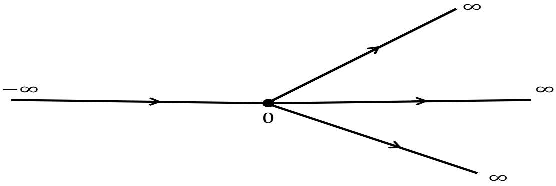
$$\mathcal{E}_q = \min_{\phi \in H^1(\mathbb{R})} E(u; \mathbb{R})$$

and the infimum is not attained. The reason is topological. By the energy-decreasing symmetry rearrangements,

$$E(u; \Gamma) > E(\hat{u}; \mathbb{R}) \geq \min_{\phi \in H^1(\mathbb{R})} E(u; \mathbb{R}) = \mathcal{E}_q.$$

A minimizing sequence escapes to infinity along an unbounded edge.

Application to the star graphs



Theorem. (Adami–Serra-Tilli, 2015)
If $N \geq 3$, no ground state exists.

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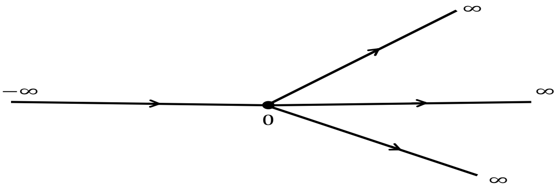

Theorem. (Adami–Serra-Tilli, 2015)
If $N \geq 3$, no ground state exists.

However, there exists a standing wave called the half-soliton:

$$\Phi(x) = \begin{bmatrix} \sqrt{|\Lambda|}\mathrm{sech}(\sqrt{|\Lambda|}x), & x \in (-\infty, 0), \; j = 1, \\ \sqrt{|\Lambda|}\mathrm{sech}(\sqrt{|\Lambda|}x), & x \in (0, \infty), \; 2 \leq j \leq N. \end{bmatrix},$$

with $\Lambda = -q^2/4$.

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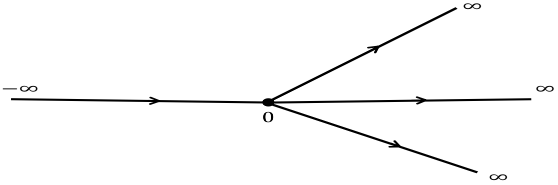

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with $\Lambda = -q^2/4$.

Theorem. (Kairzhan–P., JDE, 2018) Half-soliton is a saddle point of energy E at fixed mass Q. This saddle point is unstable in the NLS time flow.

Main goals: the limit of large mass

- Classify standing waves of NLS on a general metric graph Γ.
- Develop rigorous approximations of standing waves of NLS.
- Characterize existence of the ground state of energy.

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Theorem. (Adami–Serra–Tilli, 2019) For each finite edge e of the unbounded graph Γ, there exists a local minimizer Φ of energy E at fixed (large) mass Q such that $\|\Phi\|_{L^\infty(\Gamma)} = \|\Phi\|_{L^\infty(e)}$. Each minimizer is orbitally stable under the NLS time flow.

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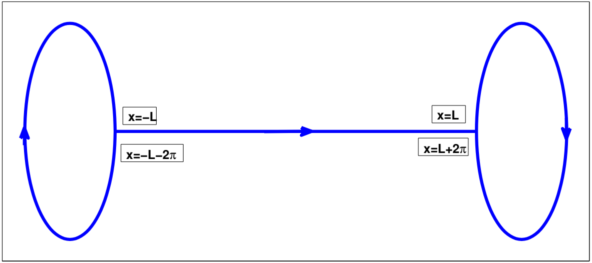
- We identify a global minimizer among these local minimizers; both for bounded and unbounded graphs.
- We work only in the cubic NLS case.
- We do not claim orbital stability of these local minimizers.

Example: Dumbbell Graph

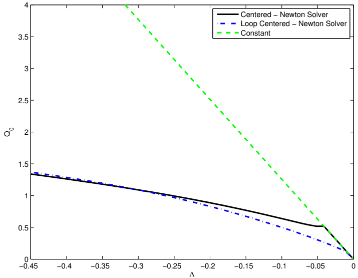


The PDE problem can be formulated in terms of components:

$$\Psi = \left[\begin{array}{ll} \psi_-(x), & x \in I_- := [-L-2\pi, -L], \\ \psi_0(x), & x \in I_0 := [-L, L], \\ \psi_+(x), & x \in I_+ := [L, L+2\pi], \end{array} \right],$$

where L is half-length of the central edge and π is half-length of the loop.

Bifurcation diagram: small mass $Q(\Psi) = q$



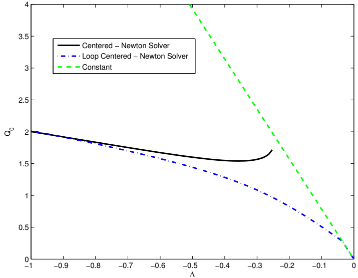


Figure: The bifurcation diagram for $L = 2\pi$ (left) and $L = \pi/2$ (right).

Symmetric state has larger mass than the asymmetric state.
The asymmetric state is the ground state of NLS on the dumbbell graph.
(Marzuola–P, 2016) (Goodman, 2018)

Bifurcation diagram: large mass $Q(\Psi) = \mu$

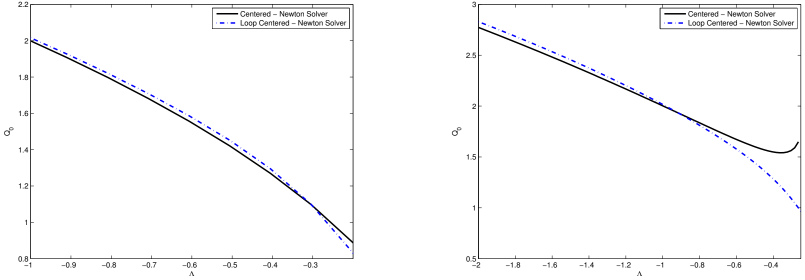
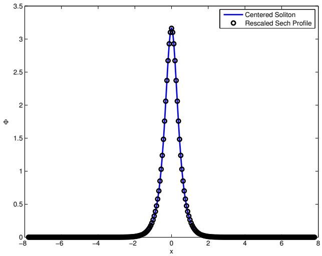


Figure: The bifurcation diagram for $L = 2\pi$ (left) and $L = \pi/2$ (right).

Symmetric state has smaller mass than the asymmetric state.
Which state is the ground state of NLS on the dumbbell graph?

Stationary states: large mass $Q(\Psi) = \mu$



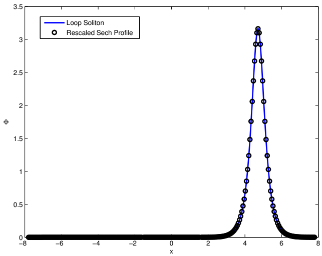


Figure: Comparison of the two stationary states (solid line) with the solitary wave (dots) for $L = \pi/2$ and $\Lambda = -10.0$.

Both stationary states are close to the NLS solitary wave:

$$\phi_\infty(x) = \sqrt{|\Lambda|}\text{sech}(\sqrt{|\Lambda|}x), \quad x \in \mathbb{R},$$

with mass $Q(\phi_\infty) = 2\sqrt{|\Lambda|}$.

Comparison Theorem in the limit of large mass

Question: Assume there exist two monotonically decreasing branches $\Lambda \mapsto \mathcal{Q}$ which satisfy

$$|\mathcal{Q}_1(\Lambda) - \mathcal{Q}_2(\Lambda)| \to 0 \quad \text{as} \quad \Lambda \to -\infty.$$

Which branch gives minimum of energy $\mathcal{E}_q$ for fixed mass $\mathcal{Q} = q$?

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Theorem (Berkolaiko–Marzuola–P, 2019)

If $\mathcal{Q}_1(\Lambda) < \mathcal{Q}_2(\Lambda)$ *for every* $\Lambda \in (-\infty, \Lambda_0)$*, then*

$$\mathcal{Q}_1(\Lambda_1) = \mathcal{Q}_2(\Lambda_2) = q \quad \Rightarrow \quad \mathcal{E}_1(\Lambda_1) > \mathcal{E}_2(\Lambda_2),$$

for every $q \gg 1$.

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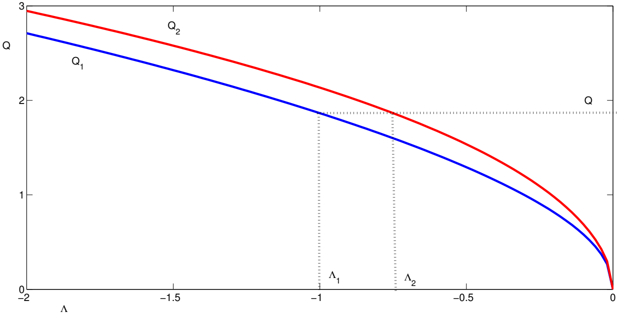
for every $q \gg 1$.

$\Rightarrow$ Asymmetric state is the ground state on the dumbbell graph.

More about the Comparison Theorem

Assume $\Phi \in H^1_\Gamma$ is a critical point of $E(u) - \Lambda Q(u)$ for the Lagrange multiplier $\Lambda < 0$. Set $\mathcal{Q}(\Lambda) = Q(\Phi)$ and $\mathcal{E}(\Lambda) = E(\Phi)$. Then,

$$\frac{d\mathcal{E}}{d\Lambda} = \Lambda \frac{d\mathcal{Q}}{d\Lambda}.$$



- If $\Lambda_1 < \Lambda_2$ and $\mathcal{Q}_2(\Lambda_2) = \mathcal{Q}_1(\Lambda_1) = q$, then $\mathcal{E}_1(\Lambda_1) > \mathcal{E}_2(\Lambda_2)$.

Numerical example: ground state in the loop for $L < \pi$

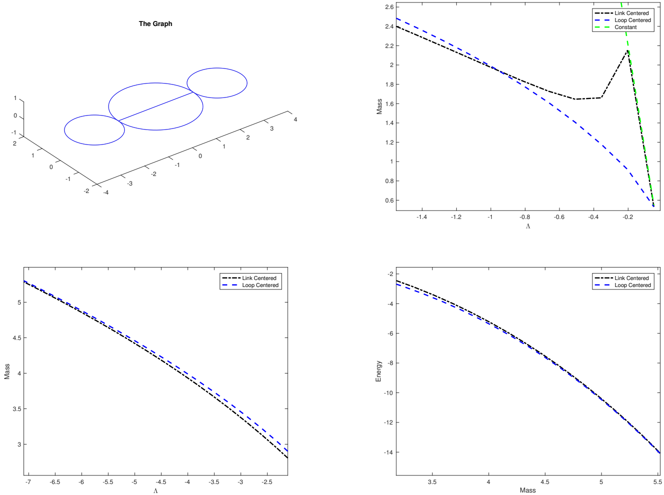
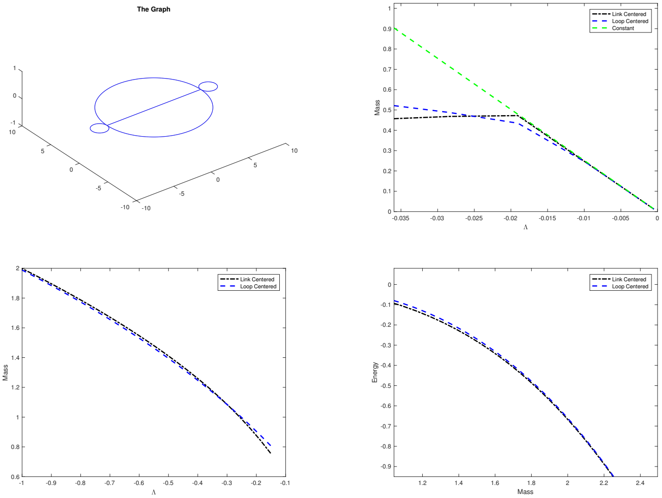


Figure: The generalized dumbbell graph (top left), the $\mathcal{Q}$ vs Λ plot bifurcating from linear theory (top right), the $\mathcal{Q}$ vs Λ plot in the large mass limit (bottom left), and the $\mathcal{E}$ vs. $\mathcal{Q}$ plot for large $\mathcal{Q}$ (bottom right).

Numerical example: ground state on the edge for $L > \pi$



Main result for bounded graphs

Theorem (Berkolaiko–Marzuola–P, 2019)

Consider a bounded graph Γ with finitely many edges of finite lengths at each vertex point. The ground state localizes at the following edge of the graph Γ:

(i) *a pendant (terminal edge) of the longest length; in case of two edges of the same longest length, a pendant with the lowest degree of the vertex.*

(ii) *If (i) is void, a loop of the shortest length connected with one edge.*

(iii) *If (i)–(ii) are void, a loop connected with two edges.*

(iv) *If (i)–(iii) are void, an edge (either a loop connected with $N \geq 3$ edges or an internal edge connected with $N_- \geq 2$ and $N_+ \geq 2$ edges) of the longest length; in case of two edges of the same length, an edge for which*

$$\frac{N-2}{N+2} \quad \text{or} \quad \sqrt{\frac{(N_- - 1)(N_+ - 1)}{(N_- + 1)(N_+ + 1)}}$$

is minimal.

Main result for unbounded graphs

Theorem (Berkolaiko–Marzuola–P, 2019)

Consider an unbounded graph Γ with finitely many edges at each vertex point with at least one edge as a half-line. The ground state exists and localizes at the following edge of the graph Γ:

(i) *a pendant (terminal edge) of the longest length; in case of two edges of the same longest length, a pendant with the lowest degree of the vertex.*

(ii) *If (i) is void, a loop of the shortest length connected with one edge.*

The ground state does not exist if the graph Γ does not have pendants or loops connected with one or two edges.

Remark: If (i)–(ii) are void but the graph Γ has a loop connected with two edges, the existence of the ground state is inconclusive at the leading order (exponentially small in μ) and needs separate consideration.

Analysis in the large mass limit

Let $\Lambda = -\mu^2 < 0$ and rescale solutions of

$$(-\Delta + \mu^2)\Phi = 2|\Phi|^2\Phi,$$

with the scaling transformation

$$\Phi(x) = \mu\Psi(z), \quad z = \mu x.$$

The stationary NLS equation becomes

$$(-\Delta + 1)\Psi = 2|\Psi|^2\Psi,$$

on the graph Γ_μ where all edge lengths are scaled by μ.

Pick an edge $e \in \Gamma_\mu$ and declare $\Gamma_\mu^c := \Gamma_\mu \backslash e$ be the rest of the graph with Neumann–Kirchhoff conditions at other vertices and only Dirichlet conditions at the boundary vertices B.

Dirichlet-to-Neumann map

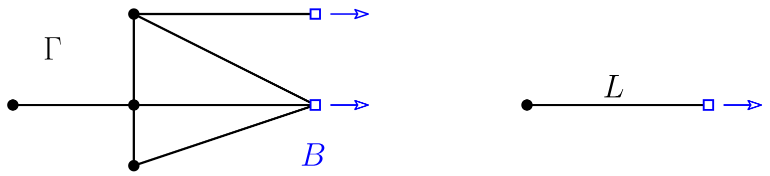


Figure: A graph with boundary vertices B marked as empty squares. Arrows indicate the outgoing derivatives of the eigenfunction in the Neumann data.

The truncated boundary-value problem:

$$\begin{cases} (-\Delta + 1)\,\Psi = 2|\Psi|^2\Psi, & \text{on every } e \in \Gamma_\mu^c, \\ \Psi \text{ satisfies NK conditions} & \text{for every } v \in V \setminus B, \\ \Psi(v_j) = p_j, & \text{for every } v_j \in B. \end{cases}$$

Neumann data is obtained from the outward derivatives:

$$q_j := \sum_{e \sim v_j} \partial u_e(v_j).$$

Small solution on Γ_μ^c

Lemma

There are $C_0 > 0$, $p_0 > 0$ and $\mu_0 > 0$ such that for every $\mathbf{p} = (p_1, \dots, p_{|B|})$ with $\|\mathbf{p}\| < p_0$ and every $\mu > \mu_0$, there exists a unique solution $\Psi \in H^2(\Gamma_\mu)$ satisfying

$$\|\Psi\|_{H^2(\Gamma_\mu)} \leq C_0 \|\mathbf{p}\|.$$

and

$$|q_j - d_j p_j| \leq C_0 \left(\|\mathbf{p}\| e^{-\mu \ell_{\min}} + \|\mathbf{p}\|^3 \right),$$

where d_j is the degree of the j-th boundary vertex and $\ell_{\min}$ is the length of the shortest edge in Γ. Moreover, the Neumann data $\mathbf{q}$ is C^1 w.r.t. $\mathbf{p}$ and μ.

Large solution on the edge e

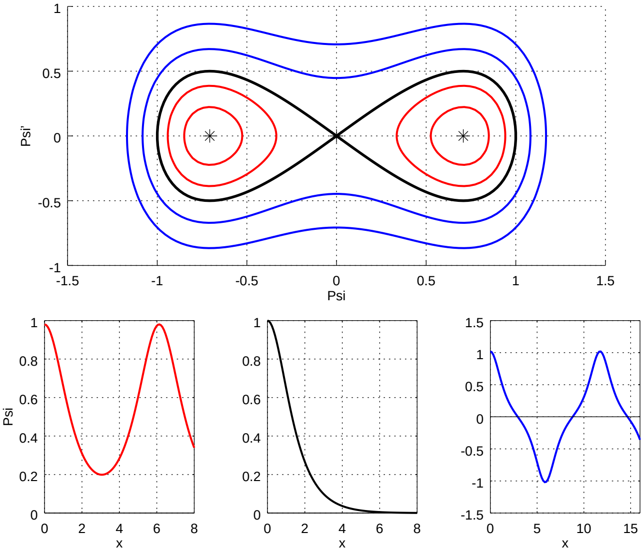


Figure: Top: phase portrait for the second-order equation $-\Psi'' + \Psi - 2\Psi^3 = 0$.
Bottom: typical solutions with initial conditions $\Psi'(0) = 0$.

Large solution on the edge e

General solution is given in terms of the elliptic functions:

$$\Psi(z) = \frac{1}{\sqrt{2-k^2}} \mathrm{dn}\left(\frac{z}{\sqrt{2-k^2}}; k\right), \quad k \in (0, \sqrt{2}).$$

Lemma

Consider an edge $[0, L]$ with the Neumann condition at $z = 0$ and the boundary vertex at $z = L = \mu\ell$. There is an interval (k_-, k_+) with

$$k_\pm = 1 \pm 8e^{-2L} + \mathcal{O}\left(Le^{-4L}\right) \quad \text{as} \quad L \to \infty,$$

such that for every $k \in (k_-, k_+)$ the solution Ψ satisfies

$$\Psi(z) > 0, \quad \Psi'(z) < 0, \quad z \in (0, L)$$

and the boundary values are given asymptotically as $L \to \infty$ by

$$\begin{cases} p_L := \Psi(L) = 2e^{-L} - \frac{1}{4}(k-1)e^{L} + \mathcal{O}\left(Le^{-3L}\right), \\ q_L := \Psi'(L) = -2e^{-L} - \frac{1}{4}(k-1)e^{L} + \mathcal{O}\left(Le^{-3L}\right). \end{cases}$$

The boundary values are C^1 functions with respect to k.

Three possible connections between Γ^c and the edge $e \in \Gamma$

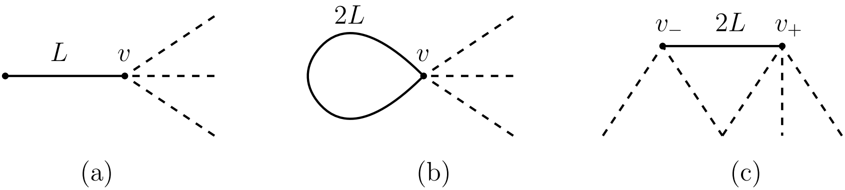


Figure: A single edge of a finite length can be connected to the remainder of the graph (shown in dashed lines) in three different ways.

For the pendant edge with the boundary vertex of degree $N+1$, we get

$$p = p_L, \quad q = -q_L, \quad L = \mu\ell.$$

Then, by the estimate on the small solution on Γ^c_μ, we get

$$-q_L = Np_L + \text{remainder}.$$

Construction of the edge-localized solutions

Lemma

The solution on the pendant edge with the boundary vertex of degree $N+1$ is described by

$$\Psi(z) = \frac{1}{\sqrt{2-k^2}} \mathrm{dn}\left(\frac{z}{\sqrt{2-k^2}}; k\right),$$

with

$$k = 1 + 8\frac{N-1}{N+1} e^{-2\mu\ell} + \mathcal{O}\left(e^{-2\mu\ell - \mu\ell_{\min}}\right),$$

where $\ell_{\min}$ is the length of the shortest edge in Γ^c. The corresponding solution $\Phi \in H^2_\Gamma$ satisfies

$$\|\Phi\|^2_{L^2(\Gamma^c)} \leq C\mu e^{-2\mu\ell}.$$

whereas the mass and energy integrals $\mathcal{Q} := Q(\Phi)$ and $\mathcal{E} := E(\Phi)$ are:

$$\begin{aligned} \mathcal{Q} &= \mu - 8\frac{N-1}{N+1}\mu^2 \ell e^{-2\mu\ell} + \mathcal{O}\left(\mu e^{-2\mu\ell}\right), \\ \mathcal{E} &= -\frac{1}{3}\mu^3 + \mathcal{O}\left(\mu^4 e^{-2\mu\ell}\right). \end{aligned}$$

Main result for bounded graphs

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Summary

Our construction of edge-localized states in the large mass limit is based on:

- Explicit solution on each edge in terms of elliptic functions.
- Surgery technique and construction of Dirichlet-to-Neumann map on the rest of the graph.
- Implicit function theorem for $-q_L = Np_L +$ remainder.
- Comparison theorem.

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- Comparison theorem.

Further remarks:

- Applications to bounded and periodic graphs are possible beyond the symmetric rearrangement theory of Adami-Serra-Tilli.
- Explicit solution in terms of elliptic functions is also available for quintic nonlinearity

Summary

Our construction of edge-localized states in the large mass limit is based on:

- Explicit solution on each edge in terms of elliptic functions.
- Surgery technique and construction of Dirichlet-to-Neumann map on the rest of the graph.
- Implicit function theorem for $-q_L = Np_L +$ remainder.
- Comparison theorem.

Further remarks:

- Applications to bounded and periodic graphs are possible beyond the symmetric rearrangement theory of Adami-Serra-Tilli.
- Explicit solution in terms of elliptic functions is also available for quintic nonlinearity

Thank you! Questions ???