Traveling waves in the Babenko equation for water waves

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Workshop "Mathematical theory of water waves",
Lund University, Sweden

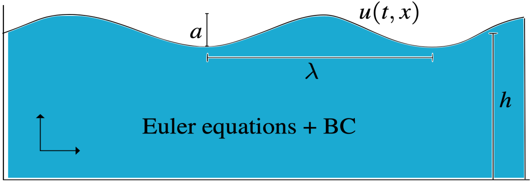
August 2024

Section 1

Background and motivations

Background and motivations

We study traveling Stokes waves in the irrotational motion of an incompressible fluid:



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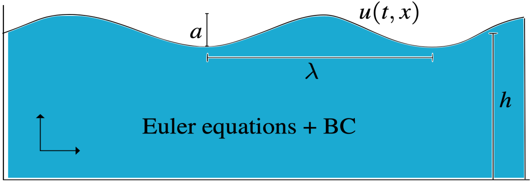

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The Korteweg–de Vries (KdV) equation:

$$u_t + u_x + u_{xxx} + u\, u_x = 0$$

[Boussinesq, 1872]   [Korteweg & de Vries, 1895]

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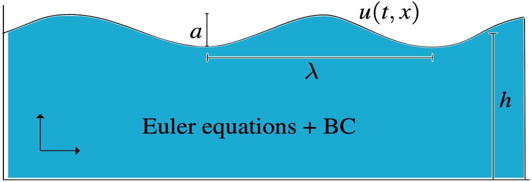

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The Benjamin–Bona–Mahony (BBM) equation

$$u_t + u_x - u_{txx} + u\,u_x = 0$$

[Peregrine, 1966] [Benjamin–Bona–Mahony, 1972]

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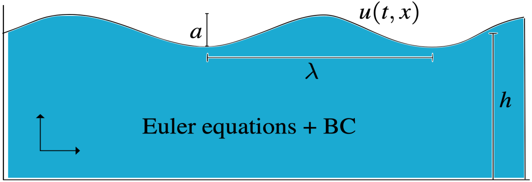


These traveling waves are approximated in the shallow limit $a \ll h \ll \lambda$ by the following local evolution equations.

The Camassa–Holm (CH) equation

$$u_t + u_x - u_{txx} + 3\,u\,u_x = 2\,u_x u_{xx} + u\,u_{xxx}$$

[Camassa & Holm, 1993] [Johnson, 2000] [Constantin & Lannes, 2009]

Traveling waves (decaying or periodic profiles)

Common features of the KdV and BBM equations:

- Solutions of the initial-value problem exist in Sobolev space H^1
- Energy, momentum, and mass are defined in H^1 and conserved
- Traveling waves $u(t,x) = U(x - ct)$ have smooth profiles U in the admissible range of the wave speed c
- Traveling waves are orbitally stable in H^1 as constrained minimizers of energy subject to fixed momentum and/or mass. Consequently, they are linearly and spectrally stable.

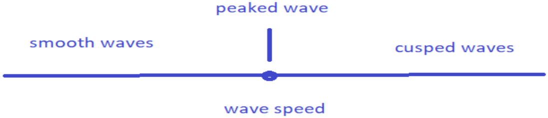
Traveling waves (decaying or periodic profiles)

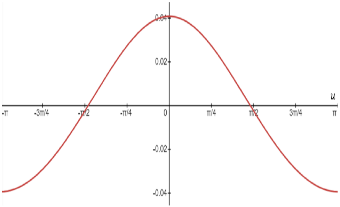
The CH equation (and CH-related models) have different properties:

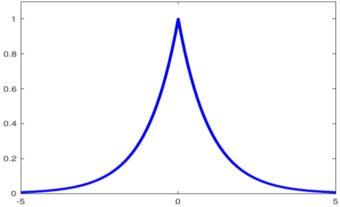
- Solutions of the initial-value problem exist in $H^1 \cap W^{1,\infty}$
  [De Lellis–Kappeler-Topalov (2007)] [Linares–Ponce–Sideris (2019)]
- Traveling waves $u(t,x) = U(x - ct)$ are smooth only in a subset of parameters and either peaked or cusped outside the subset
  [Lennels (2005)] [Geyer–Martins–Natali–P (2022)]
- Smooth and peaked waves are constrained minimizers of energy
  [Constantin & Strauss, 2000] [Constantin & Molinet, 2001] [Lennels, 2005]
- Waves with smooth profiles are stable in the time evolution
  [Constantin & Strauss, 2002] [Lennels, 2006]
- Waves with peaked profiles are unstable in the time evolution
  [Natali & P., 2020] [Madiyeva & P., 2021] [Lafortune & P., 2022]

Traveling waves (decaying or periodic profiles)

Summary on the smooth versus peaked waves

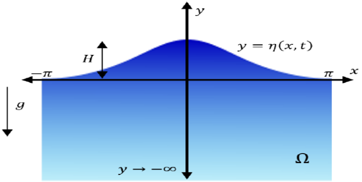


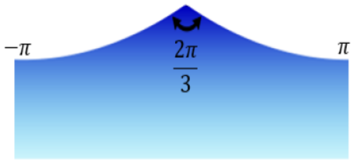




Smooth and peaked traveling Stokes waves

Stokes (1880) suggested existence of the peaked wave in the family of traveling waves:



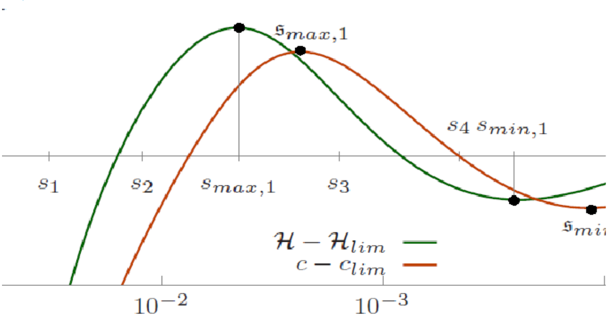


Existence of such solutions was proven by Toland (1978) and the $2\pi/3$-peaked singularity was proven by Plotnikov (1982).

Smooth and peaked traveling Stokes waves

More recently, numerical results were developed for approximation of nearly-peaked periodic waves.

[Dyachenko–Lushnikov–Korotkevich, 2016] [Lushnikov, 2016]



Instability of smooth Stokes waves was explored numerically:

[Dyachenko-Semenova, 23] [Korotkevich-Lushnikov-Semenova-Dyachenko, 23]

Smooth and peaked traveling Stokes waves

Modulation instability of small-amplitude Stokes waves was studied in a recent invasion:

- Berti–Masrepo–Ventura, 2022: by using expansions of Dirichlet-to-Neumann operator
- Creedon–Deconinck, 2023: by using expansions of the Ablowitz-Fokas-Musslimani integral formulation
- Hur–Yang, 2023: by using rescaling of the finite-depth fluid and expansions
- Nguyen–Strauss, 2023: by using complex variables and transformations

Smooth and peaked traveling Stokes waves

The objectives of our work:

- To explore a closed system of nonlinear evolution equations (similar to CH) obtained by using conformal transformations
- To investigate transition from smooth to peaked traveling waves
- To prove analytically the stability of smooth waves and the instability of peaked traveling periodic waves with respect to co-periodic perturbations.

Section 2

A closed system of nonlinear evolution equations based on Babenko equaton

Euler equations in physical coordinates

- $\eta(x,t)$ - the free surface profile.
- $\phi(x,y,t)$ - velocity potential satisfying the Laplace equation in

$$D_\eta(t) := \{(x,y) : \quad -\pi \leq x \leq \pi, \quad -h_0 \leq y \leq \eta(x,t)\}$$

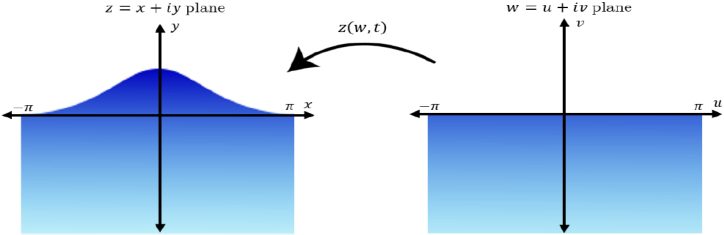
- Periodic boundary conditions at $x = \pm\pi$.
- Neumann boundary condition $\varphi_y|_{y=-h_0} = 0$.
- Nonlinear evolution equatons at the free surface:

$$\left.\begin{aligned} \eta_t + \varphi_x \eta_x - \varphi_y = 0, \\ \varphi_t + \frac{1}{2}(\varphi_x)^2 + \frac{1}{2}(\varphi_y)^2 + \eta = 0, \end{aligned}\right\} \quad \text{at} \;\; y = \eta(x,t),$$

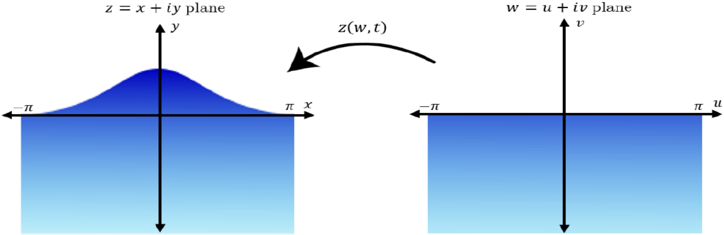
- For unique definition of h_0, we use the zero-mean constraint

$$\oint \eta dx = 0.$$

Conformal transformation



Conformal transformation



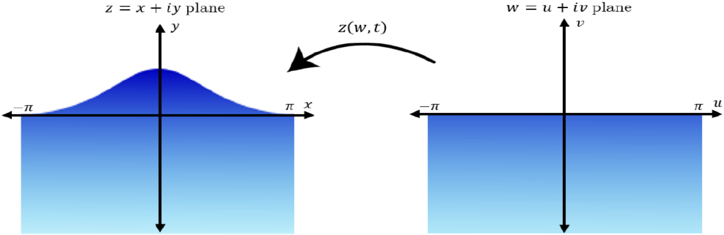
Cauchy–Riemann equations for $z = x + iy$:

$$\frac{\partial x}{\partial u} = \frac{\partial y}{\partial v}, \qquad \frac{\partial x}{\partial v} = -\frac{\partial y}{\partial u}$$

$$\text{in} \quad \mathcal{D} := \{(u, v) : \quad -\pi \le u \le \pi, \quad -h \le v \le 0\}$$

subject to Neumann condition $\partial_v x|_{v=-h} = 0$ due to $y(u, -h, t) = -h_0$.

Conformal transformation

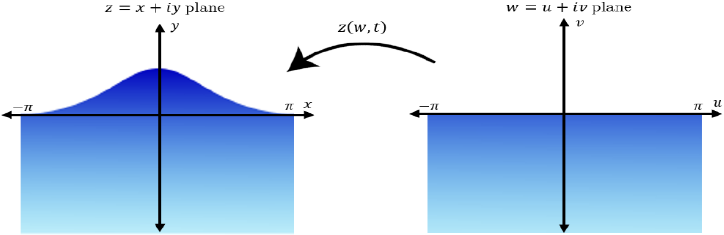


Fourier series solution:

$$x(u,v,t) = u + \sum_{n\in\mathbb{Z}} \hat{x}_n(t) e^{inu} \, \frac{\cosh(n(v+h))}{\cosh(nh)},$$

$$y(u,v,t) = v + h - h_0 + \sum_{n\in\mathbb{Z}} \hat{x}_n(t) e^{inu} \, i \, \frac{\sinh(n(v+h))}{\cosh(nh)}.$$

Conformal transformation



The velocity potential is then uniquely represented by

$$\varphi(u,v,t) = \sum_{n\in\mathbb{Z}} \hat{\xi}_n(t) e^{inu} \frac{\cosh(n(v+h))}{\cosh(nh)},$$

where $\hat{\xi}_n(t)$ is the Fourier coefficient for $\xi(u,t) = \varphi(u, v=0, t)$. The other canonical variable is $\eta(u,t) = y(u, v=0, t)$.

Evolution equations for $\xi(u,t)$ and $\eta(u,t)$

The closed system of two evolution equations is

$$\left\{ \begin{array}{l} (1+K_h\eta)\eta_t - \eta_u T_h^{-1}\eta_t + T_h\xi_u = 0, \\ \xi_t\eta_u - \xi_u\eta_t + \eta\eta_u + T_h\left[(1+K_h\eta)\xi_t - \xi_u T_h^{-1}\eta_t + (1+K_h\eta)\eta\right] = 0, \end{array} \right.$$

where skew-adjoint operators T_h and T_h^{-1} are defined by

$$\widehat{(T_h)}_n = i\tanh(hn), \quad n \in \mathbb{Z}, \quad \widehat{\left(T_h^{-1}\right)}_n = \left\{ \begin{array}{ll} -i\coth(hn), & n \in \mathbb{Z}\backslash\{0\}, \\ 0, & n = 0, \end{array} \right.$$

whereas the self-adjoint operator $K_h = T_h^{-1}\partial_u$ is defined by

$$\widehat{(K_h)}_n = \left\{ \begin{array}{ll} n\coth(hn), & n \in \mathbb{Z}\backslash\{0\}, \\ 0, & n = 0. \end{array} \right.$$

[Dyachenko-elder–Kuznetsov–Spector–Zakharov, 1996]

[Dyachenko-junior–Lushnikov–Korotkevich, 2016]

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Since $\partial_u x(u,0,t) = 1 + K_h\eta$, the original constraint $\oint \eta dx = 0$ becomes

$$\oint \eta(1+K_h\eta)du = 0.$$

Additional constants of motion are

$$\oint \xi\eta_u du, \quad \oint \xi(1+K_h\eta)du, \quad \oint \left[\eta^2(1+K_h\eta) - \xi T_h\xi_u\right] du.$$

These are the horizontal and vertical momenta and the energy.

[Benjamin–Olver, 1982]

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Traveling waves $\eta(u,t) = \eta(u-ct)$ satisfy $\xi = cT_h^{-1}\eta$, where the profile η is a solution of Babenko's equation [Babenko, 1987]

$$\boxed{(c^2K_h - 1)\eta = \frac{1}{2}K_h\eta^2 + \eta K_h\eta.}$$

Both smooth and peaked traveling waves are solutions of this scalar equation. Their linear stability is related to the linearized operator

$$\boxed{\mathcal{L}_h v := (c^2K_h - 1)v - K_h\eta v - vK_h\eta - \eta K_h v}$$

which is self-adjoint in $L^2_{\rm per}(\mathbb{T})$.

Section 3

Existence results for the deep water: $h \to \infty$

Babenko's equation

Traveling waves $\eta(u,t) = \eta(u - ct)$ satisfy Babenko's equation:

$$\boxed{(c^2 K_h - 1)\eta = \frac{1}{2} K_h \eta^2 + \eta K_h \eta,}$$

where

$$\widehat{(K_h)}_n = \begin{cases} n \coth(hn), & n \in \mathbb{Z}\backslash\{0\}, \\ 0, & n = 0. \end{cases}$$

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In the deep water limit $h \to \infty$, $K_h \to \partial_u H$, where H is the Hilbert transform on 2π-periodic functions:

$$f = \sum_{n \in \mathbb{Z}} f_n e^{inu} \quad \Rightarrow \quad Hf = \sum_{n \in \mathbb{Z}} (-i)\mathrm{sgn}(n) f_n e^{inu}.$$

Babenko's equation in the deep water limit

We have the main model:

$$\boxed{c^2 H\partial_u\eta - \eta = H(\eta\partial_u\eta) + \eta H\partial_u\eta,}$$

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$$\boxed{c^2 H\partial_u \eta - \eta = H(\eta \partial_u \eta) + \eta H \partial_u \eta,}$$

Small-amplitude (Stokes) expansions are algorithmically computed:

$$\eta(u) = a\cos(u) + a^2\left[\cos(2u) - \frac{1}{2}\right] + \frac{3}{2}a^3\cos(3u) + \mathcal{O}(a^4)$$

and

$$c^2 = 1 + a^2 + \mathcal{O}(a^4),$$

where $a > 0$ is a small parameter for the wave amplitude.

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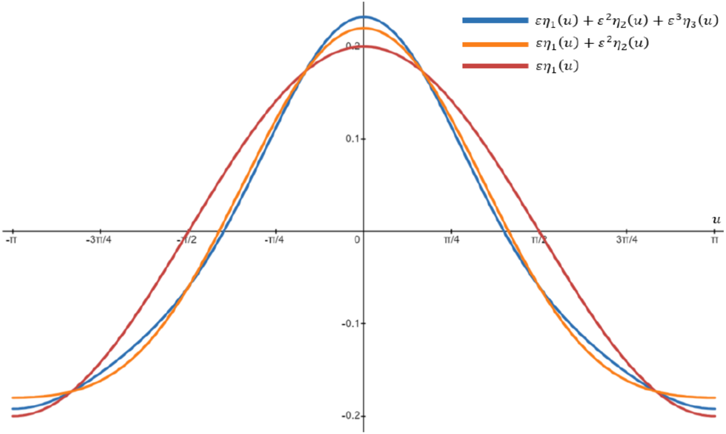

Babenko's equation in the deep water limit

Near the singular waves, it makes sense to use $\eta(u) = \frac{c^2}{2} - \zeta(u)$ with ζ satisfying the fixed-point equation

$$\zeta = T(\zeta) := H(\zeta \partial_u \zeta) + \zeta H \partial_u \zeta + \frac{c^2}{2},$$

with the “boundary" conditions $\zeta(0) = 0$ and $\partial_u \zeta(\pm\pi) = 0$.

Babenko's equation in the deep water limit

$$\boxed{\zeta = H(\zeta \partial_u \zeta) + \zeta H \partial_u \zeta + \frac{c^2}{2}}$$

Theorem (Locke–P, 2024)

If the solution of $\zeta = T(\zeta)$ *is singular at* $u = 0$ *with the singularity of the type*

$$\zeta(u) = A|u|^\alpha + \mathcal{O}(|u|^{2\alpha}), \quad \alpha \in (0, 1],$$

with some $A > 0$, *then necessarily,* $\alpha = \frac{2}{3}$.

In agreement with Stokes (1880), Toland (1978), Plotnikov (1982).

Babenko's equation in the deep water limit

Theorem (Locke–P, 2024)

If the solution of $\zeta = T(\zeta)$ is singular at $u = 0$ with the singularity of the type

$$\zeta(u) = A|u|^{2/3} + B|u|^{\beta} + \mathcal{O}(|u|^{2/3+\beta}), \quad \beta \in \left(\frac{2}{3}, 2\right),$$

with some $A > 0$ and $B \neq 0$, then necessarily, $\beta \approx 1.46$ is a root of the transcendental equation

$$\left(\beta + \frac{2}{3}\right) \cot\left(\frac{\pi}{2}(\beta - \frac{1}{3})\right) - \beta \tan\left(\frac{\pi\beta}{2}\right) = \frac{2}{\sqrt{3}}.$$

In agreement with Grant (1973).

Parameters c, A, B are not defined by the local expansion.

Towards stability analysis

The linearized Babenko's operator is

$$\boxed{\mathcal{L}_\infty \varphi := (c^2 H\partial_u - 1)\varphi - H\partial_u(\eta\varphi) - (H\partial_u\eta)\varphi - \eta H\partial_u\varphi.}$$

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Recall the small-amplitude expansion:

$$\eta(u) = a\cos(u) + a^2\left[\cos(2u) - \frac{1}{2}\right] + \frac{3}{2}a^3\cos(3u) + \mathcal{O}(a^4),$$
$$c^2 = 1 + a^2 + \mathcal{O}(a^4),$$

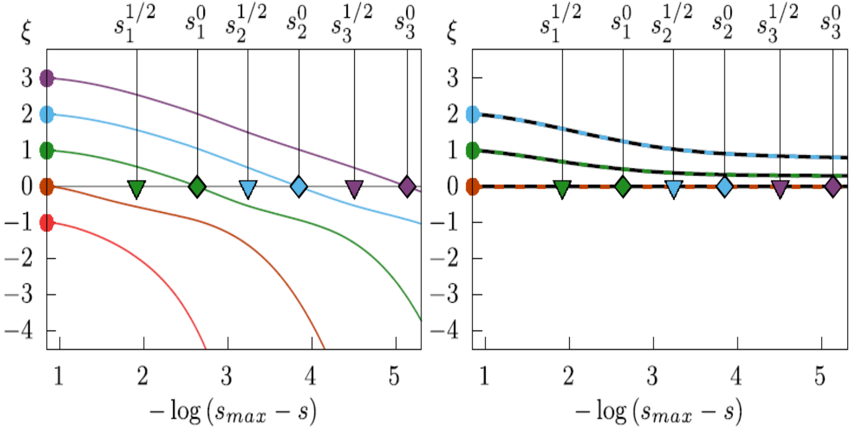
where $a > 0$ is the small-amplitude parameter. Then, we know the spectrum of $\mathcal{L}_\infty$ for $a = 0$ and for small $a > 0$:

$$a = 0: \quad \sigma(\mathcal{L}_\infty) = \{|n| - 1, \quad n \in \mathbb{Z}\} = \{-1, 0, 1, 2, \dots\}.$$

The zero EV splits into a zero eigenvalue and a small negative EV $-2a^2 + \mathcal{O}(a^4)$ in agreement with Dyachenko–Semenova (2023)

Towards stability analysis

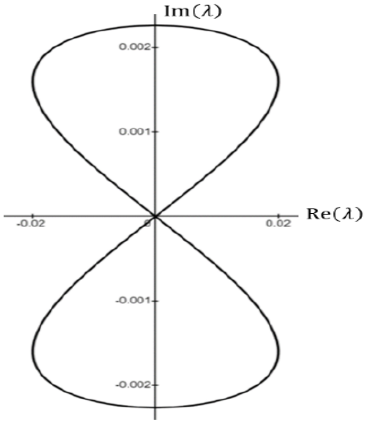
Numerical results from Dyachenko–Semenova (2023)



Work in progress: the stability criterion from the energy

Towards stability analysis

The splitting of zero eigenvalue of $\mathcal{L}$ induces the figure-eight modulational instability:



in agreement with Berti (2022), Creedon–Deconinck (2023), others.

Section 4

Toy model for shallow water waves

Evolution equation for $\eta(u,t)$

Full system of evolution equations:

$$\left\{ \begin{array}{l} (1+K_h\eta)\eta_t - \eta_u T_h^{-1}\eta_t + T_h\xi_u = 0, \\ \xi_t\eta_u - \xi_u\eta_t + \eta\eta_u + T_h\left[(1+K_h\eta)\xi_t - \xi_u T_h^{-1}\eta_t + (1+K_h\eta)\eta\right] = 0, \end{array} \right.$$

If $\eta(u,t) = \eta(u-ct,t)$ and $\xi = cT_h^{-1}\eta + \zeta$, the system can be simplified into the form:

$$(1+K_h\eta)\eta_t - \eta_u T_h^{-1}\eta_t + T_h\zeta_u = 0$$

and

$$(1+K_h\eta)\zeta_t - \zeta_u T_h^{-1}\eta_t + T_h^{-1}(\zeta_t\eta_u - \zeta_u\eta_t)$$
$$\boxed{+2cT_h^{-1}\eta_t - c^2K_h\eta + (1+K_h\eta)\eta + \frac{1}{2}K_h\eta^2 = 0.}$$

Evolution equation for $\eta(u,t)$

We consider the scalar evolution equation:

$$\boxed{2cT_h^{-1}\eta_t - c^2 K_h \eta + (1 + K_h \eta)\eta + \frac{1}{2} K_h \eta^2 = 0.}$$

where

$$\widehat{\left(T_h^{-1}\right)}_n = \left\{ \begin{array}{c} -i\coth(hn), \\ 0, \end{array} \right. \qquad \widehat{(K_h)}_n = \left\{ \begin{array}{cc} n\coth(hn), & n \in \mathbb{Z}\backslash\{0\}, \\ 0, & n = 0. \end{array} \right.$$

Evolution equation for $\eta(u,t)$

We consider the scalar evolution equation:

$$\boxed{2cT_h^{-1}\eta_t - c^2 K_h \eta + (1 + K_h\eta)\eta + \frac{1}{2}K_h\eta^2 = 0.}$$

Recall the intermediate long–wave (ILW) equation (integrable PDE)

$$\partial_t \eta + h^{-1}\partial_u \eta + \eta \partial_u \eta = \mathcal{K}_h(\partial_u \eta)$$

where

$$\mathcal{K}_h = K_h + \frac{1}{2\pi h}\oint \cdot \, du, \quad \widehat{(\mathcal{K}_h)}_n = \left\{ \begin{array}{ll} n\coth(hn), & n \in \mathbb{Z}\backslash\{0\}, \\ h^{-1}, & n = 0. \end{array} \right.$$

As $h \to 0$, $\mathcal{K}_h = h^{-1} - \frac{1}{3}h\partial_u^2 + \mathcal{O}(h^3)$ and the ILW equation converges to the KdV equation after rescaling.

Evolution equation for $\eta(u,t)$

We consider the scalar evolution equation:

$$\boxed{2cT_h^{-1}\eta_t - c^2 K_h \eta + (1 + K_h\eta)\eta + \frac{1}{2}K_h\eta^2 = 0.}$$

This promts us to consider K_h replaced by

$$\widehat{(\tilde{K}_h)}_n = \begin{cases} n\coth(hn) - h^{-1}, & n \in \mathbb{Z}\backslash\{0\}, \\ 0, & n = 0. \end{cases}$$

As $h \to 0$, $\tilde{K}_h = -\frac{1}{3}h\partial_u^2 + \mathcal{O}(h^3)$ and the evolution equation for $\eta(u,t)$ converges to the new local model after rescaling:

$$\boxed{2c\partial_u\partial_t\eta = (c^2 - 2\eta)\partial_u^2\eta - (\partial_u\eta)^2 + \eta.}$$

This is the Hunter–Saxton equation derived in a different context.

Conserved quantities of the toy model

Thus, we can consider the toy model in the form:

$$\boxed{2c\partial_u\partial_t\eta = (c^2 - 2\eta)\partial_u^2\eta - (\partial_u\eta)^2 + \eta}$$

The toy model has the same constraint

$$\oint \left[\eta + (\partial_u\eta)^2\right] du = 0$$

and the same conserved quantities

$$\oint \eta du, \quad \oint (\partial_u\eta)^2 du, \quad \oint \left[\eta^2 + 2\eta(\partial_u\eta)^2\right] du$$

as the original system of evolution equations (but local).

Section 5

Main results on the toy model

Local well-posedness of the initial-value problem

The toy model

$$\boxed{2c\partial_u\partial_t\eta = (c^2 - 2\eta)\partial_u^2\eta - (\partial_u\eta)^2 + \eta}$$

can be rewritten as the evolution equation

$$2c\partial_t\eta = (c^2 - 2\eta)\partial_u\eta + \Pi_0\partial_u^{-1}\Pi_0\left[(\partial_u\eta)^2 + \eta\right]$$

subject to the constraint $\oint\left[\eta + (\partial_u\eta)^2\right]du = 0$. The inviscid Burgers equation

$$2c\partial_t\eta = (c^2 - 2\eta)\partial_u\eta$$

is locally well-posed in $H^1_{\rm per}(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T})$ and the mapping

$$\Pi_0\partial_u^{-1}\Pi_0\left[(\partial_u\eta)^2 + \eta\right] : H^1_{\rm per}(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T}) \to H^1_{\rm per}(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T})$$

is bounded on every bounded subset.

Local well-posedness of the initial-value problem

The toy model

$$\boxed{2c\partial_u\partial_t\eta = (c^2 - 2\eta)\partial_u^2\eta - (\partial_u\eta)^2 + \eta}$$

can be rewritten as the evolution equation

$$2c\partial_t\eta = (c^2 - 2\eta)\partial_u\eta + \Pi_0\partial_u^{-1}\Pi_0\left[(\partial_u\eta)^2 + \eta\right]$$

subject to the constraint $\oint\left[\eta + (\partial_u\eta)^2\right]du = 0$.

By standard technique (e.g. via characteristics), we obtain

Theorem

The initial-value problem is locally well-posed in $H^1_{\mathrm{per}}(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T})$.

Existence of the periodic wave solutions

If $\eta(u,t) = \eta(u)$ in the traveling wave frame, then η is a solution of the differential equation

$$\boxed{(c^2 - 2\eta)\eta'' - (\eta')^2 + \eta = 0, \qquad u \in \mathbb{T}.}$$

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Theorem

There exist $c_* := \frac{\pi}{2\sqrt{2}}$ *and* $c_\infty \in (c_*, \infty)$ *such that the ODE admits a unique solution with the profile* $\eta \in C^\infty_{\rm per}(\mathbb{T})$ *for every* $c \in (1, c_*)$ *s.t.*

$$\|\eta\|_{L^\infty} \to 0 \quad \text{as} \quad c \to 1$$

and a solution with the profile $\eta \in C^0_{\rm per}(\mathbb{T})$ *for every* $c \in (c_*, c_\infty)$ *satisfying for some* $A(c) > 0$*,*

$$\eta(u) = \frac{c^2}{2} - A(c)|u|^{2/3} + \mathcal{O}(|u|) \quad \text{as} \quad u \to 0.$$

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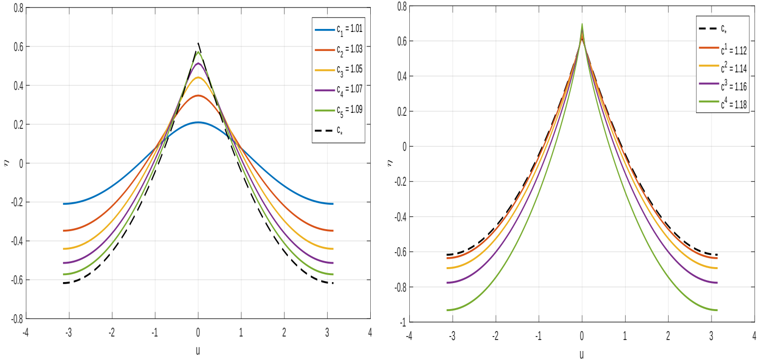
$$(c^2 - 2\eta)\eta'' - (\eta')^2 + \eta = 0, \qquad u \in \mathbb{T}.$$



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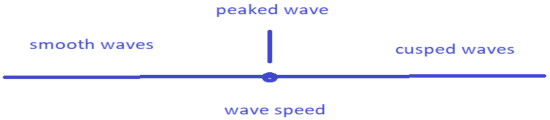
The two continuous families meet at $c = c_*$, where the profile $\eta \in C^0_{\rm per}(\mathbb{T}) \cap W^{1,\infty}(\mathbb{T})$ is explicit:

$$\eta(u) = \frac{1}{16}(\pi^2 - 4\pi|u| + 2u^2), \qquad u \in \mathbb{T}.$$



Existence of the periodic wave solutions

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Interesting that the highest amplitude

$$\max_{u \in \mathbb{T}} \eta(u) = \eta(0) = \frac{c^2}{2}$$

follows from laws of hydrodynamics and that the $|u|^{2/3}$ singularity corresponds after the conformal transformation to Stokes' law of the 120^0 angle in the physical coordinate.

The peaked profile at $c = c_*$ might be an artefact of the local model.

Tools for existence analysis

Smooth solutions of $\boxed{(c^2 - 2\eta)\eta'' - (\eta')^2 + \eta = 0}$ are level curves of $E(\eta, \eta') := \frac{1}{2}(c^2 - 2\eta)(\eta')^2 + \frac{1}{2}\eta^2$ on the phase plane (η, η').

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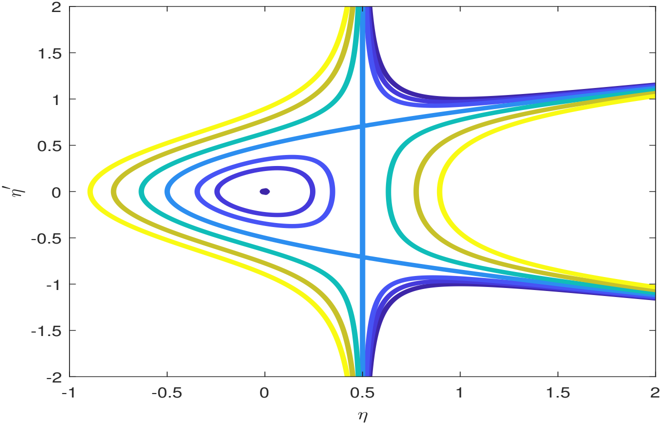

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For smooth periodic solutions, we can introduce the period function

$$T(\mathcal{E}, c) := 2\int_{-\sqrt{2\mathcal{E}}}^{\sqrt{2\mathcal{E}}} \frac{\sqrt{c^2 - 2\eta}}{\sqrt{2\mathcal{E} - \eta^2}} d\eta, \qquad \mathcal{E} \in (0, \mathcal{E}_c),$$

such that $\eta(u + T(\mathcal{E}, c)) = \eta(u)$.

Theorem

For every $c > 0$ and $\mathcal{E} \in (0, \mathcal{E}_c)$ with $\mathcal{E}_c := \frac{c^4}{8}$,

$$\partial_c T(\mathcal{E}, c) > 0 \quad \text{and} \quad \partial_{\mathcal{E}} T(\mathcal{E}, c) < 0.$$

There exists a unique root of $T(\mathcal{E}, c) = 2\pi$ for $\mathcal{E}$ in $(0, \mathcal{E}_c)$.

Tools for existence analysis

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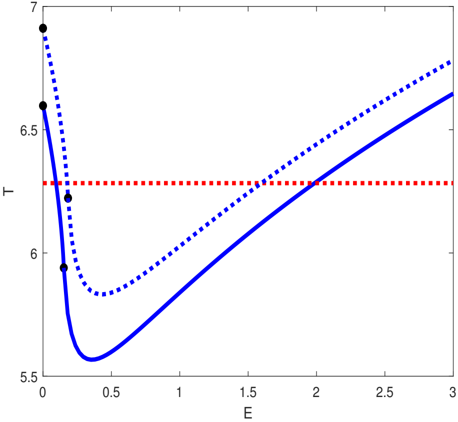

Tools for existence analysis

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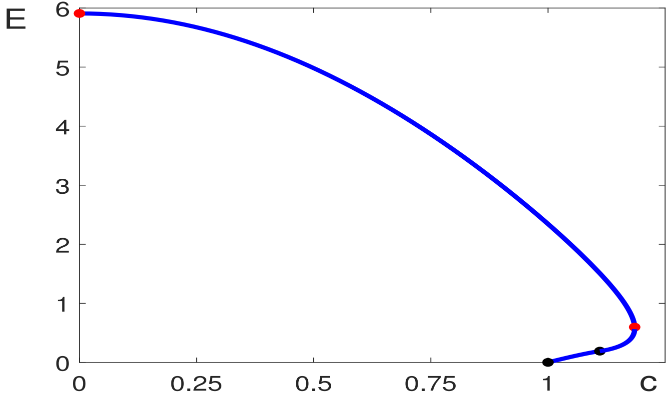

Linear stability of periodic waves with smooth profile

Starting with

$$2c\partial_t\eta = (c^2 - 2\eta)\partial_u\eta + \partial_u^{-1}\left[(\partial_u\eta)^2 + \eta\right]$$

we set $\eta(u) + \upsilon(u,t)$ and linearize at υ:

$$2c\partial_t\upsilon = -\partial_u^{-1}\mathcal{L}\upsilon, \qquad \mathcal{L} = -\partial_u(c^2 - 2\eta)\partial_u - 1 + 2\eta''.$$

The constraint $\oint\left[\eta + (\partial_u\eta)^2\right]du = 0$ yields $\langle 1 - 2\eta'', \upsilon\rangle = 0$ on the perturbation υ. Moreover, the constraints $\langle 1, \upsilon\rangle = 0$ and $\langle \eta'', \upsilon\rangle = 0$ persist in time t. They correspond to the requirement that the conserved quantities

$$\oint \eta du \quad \text{and} \quad \oint (\partial_u\eta)^2 du$$

do not change under the perturbation υ at the linear order.

Linear stability of periodic waves with smooth profile

Theorem

Consider the unique solution with the profile $\eta \in C^\infty_{\rm per}(\mathbb{T})$ *for* $c \in (1, c_*)$. *For every initial data* $\upsilon_0 \in H^1_{\rm per}(\mathbb{T})$ *satisfying* $\langle 1, \upsilon_0 \rangle = 0$ *and* $\langle \eta'', \upsilon_0 \rangle = 0$, *there exists a unique solution* $\upsilon \in C^0(\mathbb{R}, H^1_{\rm per}(\mathbb{T}))$ *of the linearized equation*

$$2c\partial_t \upsilon = -\partial_u^{-1} \mathcal{L} \upsilon, \qquad \mathcal{L} = -\partial_u (c^2 - 2\eta) \partial_u - 1 + 2\eta''.$$

with $\upsilon|_{t=0} = \upsilon_0$ *and a unique* $a \in C^0(\mathbb{R}, \mathbb{R})$ *such that*

$$\|\upsilon(\cdot, t) - a(t)\eta'\|_{H^1_{\rm per}} \leq C \|\upsilon_0\|_{H^1_{\rm per}}, \quad |a'(t)| \leq C \|\upsilon_0\|_{H^1_{\rm per}}, \quad t \in \mathbb{R},$$

where $C > 0$ *is independent of* υ_0.

Linear stability of periodic waves with smooth profile

- Linear stability implies spectral stability in the sense that there exist no solutions of the spectral problem

$$\partial_u^{-1} \mathcal{L} \upsilon_0 = \lambda_0 \upsilon_0, \qquad \eta_0 \in H^1_{\rm per}(\mathbb{T})$$

  for $\lambda \notin i\mathbb{R}$, that is, $\sigma(\partial_u^{-1}\mathcal{L}) \subset i\mathbb{R}$. Interesting that the spectral problem $\mathcal{L}\upsilon = \lambda \partial_u \upsilon$ has been considered before in [Stanislovova–Stefanov, 2016] .

- Linear stability does not imply nonlinear stability because we have no local well-posedness in $H^1_{\rm per}(\mathbb{T})$ but the $W^{1,\infty}$-norm of the perturbation υ is not controlled in the time evolution.

- The peaked wave at $c = c_*$ is likely unstable since it is similar to the other fluid models: [Geyer–P, 2020] [Lafortune–P, 2022].

- Nothing is known on cusped waves.

Tools for stability analysis

We analyze the linearized equation

$$\boxed{2c\partial_t \upsilon = -\partial_u^{-1}\mathcal{L}\upsilon, \qquad \mathcal{L} = -\partial_u(c^2 - 2\eta)\partial_u - 1 + 2\eta''}$$

associated with skew-adjoint ∂_u^{-1} and self-adjoint $\mathcal{L}$ in $L^2(\mathbb{T})$.

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We want to use the energy quadratic form in $H^1_{\rm per}(\mathbb{T})$:

$$\langle \mathcal{L}\upsilon, \upsilon \rangle = \oint \left[(c^2 - 2\eta)(\partial_u \upsilon)^2 + (2\eta'' - 1)\upsilon^2 \right] du,$$

which is constant in time t under the linear evolution.

Tools for stability analysis

We analyze the linearized equation

$$2c\partial_t \upsilon = -\partial_u^{-1}\mathcal{L}\upsilon, \qquad \mathcal{L} = -\partial_u(c^2 - 2\eta)\partial_u - 1 + 2\eta''$$

associated with skew-adjoint ∂_u^{-1} and self-adjoint $\mathcal{L}$ in $L^2(\mathbb{T})$.

For the self-adjoint operator $\mathcal{L} : H^2_{\rm per}(\mathbb{T}) \to L^2(\mathbb{T})$.

- The spectrum $\sigma(\mathcal{L})$ consists of isolated eigenvalues.
- We have $0 \in \sigma(\mathcal{L})$ because $\mathcal{L}\eta' = 0$ and 0 is a simple eigenvalue because $\partial_{\mathcal{E}}\eta$ is not 2π-periodic.
- There exist two negative eigenvalues in $\sigma(\mathcal{L})$ because $\partial_{\mathcal{E}} T(\mathcal{E}, c) < 0$.

Tools for stability analysis

We analyze the linearized equation

$$\boxed{2c\partial_t v = -\partial_u^{-1}\mathcal{L}v, \qquad \mathcal{L} = -\partial_u(c^2 - 2\eta)\partial_u - 1 + 2\eta''}$$

associated with skew-adjoint ∂_u^{-1} and self-adjoint $\mathcal{L}$ in $L^2(\mathbb{T})$.

For the constrained self-adjoint operator $\mathcal{L}|_{\mathcal{X}_c}$ due to two constraints $\langle 1, v\rangle = \langle \eta'', v\rangle = 0$, we use $\mathcal{L}\partial_c\eta = 2c\eta''$ and $\mathcal{L}1 = 2\eta'' - 1$ to compute

$$A = \begin{bmatrix} \langle \mathcal{L}^{-1}1, 1\rangle & \langle \mathcal{L}^{-1}1, \eta''\rangle \\ \langle \mathcal{L}^{-1}\eta'', 1\rangle & \langle \mathcal{L}^{-1}\eta'', \eta''\rangle \end{bmatrix} = \begin{bmatrix} c^{-1}\langle \partial_c\eta, 1\rangle - 2\pi & (2c)^{-1}\langle \partial_c\eta, 1\rangle \\ (2c)^{-1}\langle \partial_c\eta, 1\rangle & (4c)^{-1}\langle \partial_c\eta, 1\rangle \end{bmatrix}.$$

A has two negative eigenvalues and $\mathcal{L}|_{\mathcal{X}_c}$ has no negative eigenvalues and a simple zero eigenvalue if and only if the mapping $c \mapsto \mathcal{M}(c) := \oint \eta du$ is monotonically decreasing at $c \in (1, c_*)$.

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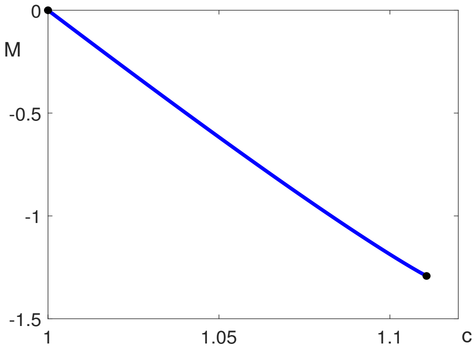

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associated with skew-adjoint ∂_u^{-1} and self-adjoint $\mathcal{L}$ in $L^2(\mathbb{T})$.

Since $\langle \mathcal{L}\upsilon, \upsilon \rangle$ is a conserved quantity for the linearized evolution, we have for $\upsilon(\cdot,t) = a(t)\eta' + w(\cdot,t)$ with $\langle \eta', w(\cdot,t) \rangle = 0$ that

$$\begin{aligned}
\alpha \|w(\cdot,t)\|^2_{H^1_{\rm per}} &\leq \langle \mathcal{L} w(\cdot,t), w(\cdot,t) \rangle \\
&= \langle \mathcal{L} \upsilon(\cdot,t), \upsilon(\cdot,t) \rangle \\
&= \langle \mathcal{L} \upsilon_0, \upsilon_0 \rangle \\
&\leq \beta \|\upsilon_0\|^2_{H^1_{\rm per}},
\end{aligned}$$

which yields $\|\upsilon(\cdot,t) - a(t)\eta'\|_{H^1_{\rm per}} \leq C \|\upsilon_0\|_{H^1_{\rm per}}$.

Summary

We have introduced here a new toy model for gravity water waves:

$$\boxed{2cT_h^{-1}\eta_t - c^2 K_h \eta + (1 + K_h \eta)\eta + \frac{1}{2} K_h \eta^2 = 0,}$$

in the context of Euler equations in holomorphic coordinates.

- It is the exact system of equations for traveling waves.
- It is a first-order reduction of Euler's equations for the linearized stability and the nonlinear evolution problems.

Summary

We have introduced here a new toy model for gravity water waves:

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in the context of Euler equations in holomorphic coordinates.

In the shallow water limit, our principal results are:

- The continuous families of smooth and cusped waves are connected at a single peaked wave.
- The smooth waves are linearly stable in the time evolution.
- The peaked wave is linearly unstable in the time evolution.
- The initial-value problem is locally well-posed in $H^1 \cap W^{1,\infty}$, which excludes the cusped waves.