Vortices in a harmonic potential

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References:

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Introduction

Density waves in cigar–shaped Bose–Einstein condensates with repulsive inter-atomic interactions and a harmonic potential are modeled by the Gross-Pitaevskii equation

$$iv_\tau = -\frac{1}{2}\nabla_\xi^2 v + \frac{1}{2}|\xi|^2 v + |v|^2 v - \mu v,$$

where μ is the chemical potential, $\xi \in \mathbb{R}^d$, and ∇_ξ^2 is the Laplacian in ξ.

Using the scaling transformation,

$$v(\xi, t) = \mu^{1/2} u(x, t), \quad \xi = (2\mu)^{1/2} x, \quad \tau = 2t,$$

the Gross–Pitaevskii equation is transformed to the semi-classical form

$$i\,\varepsilon\, u_t + \varepsilon^2\, \nabla_x^2 u + (1 - |x|^2 - |u|^2) u = 0,$$

where $\varepsilon = (2\mu)^{-1}$ is a small parameter.

Ground state

Limit $\mu \to \infty$ or $\varepsilon \to 0$ is referred to as the semi-classical or Thomas–Fermi limit. Physically, it is the limit of large density of the atomic cloud.

Let η_ε be the real positive solution of the stationary problem (ground state)

$$\varepsilon^2 \nabla_x^2 \eta_\varepsilon + (1 - |x|^2 - \eta_\varepsilon^2)\eta_\varepsilon = 0, \quad x \in \mathbb{R}^d,$$

where d is either 1, 2, or 3.

Theorem (Ignat & Milot, JFA (2006))

For sufficiently small $\varepsilon > 0$, there exists a global minimizer of the Gross–Pitaevskii energy

$$E_\varepsilon(u) = \int_{\mathbb{R}^d} \left(\frac{1}{2}\varepsilon^2 |\nabla_x u|^2 + \frac{1}{2}(|x|^2 - 1)|u|^2 + \frac{1}{4}|u|^4 \right) dx$$

in the energy space

$$\mathcal{H}_1 = \left\{ u \in H^1(\mathbb{R}^d) : \ |x|u \in L^2(\mathbb{R}^d) \right\}.$$

Ground state in the asymptotic theory

For small $\varepsilon > 0$, the ground state $\eta_\varepsilon \in \mathcal{C}^\infty(\mathbb{R})$ decays to zero as $|x| \to \infty$ faster than any exponential function

$$0 < \eta_\varepsilon(x) \leq C\,\varepsilon^{1/3} \exp\left(\frac{1-|x|^2}{4\,\varepsilon^{2/3}}\right), \quad \text{for all } |x| \geq 1.$$

The Thomas–Fermi approximation is

$$\eta_0(x) := \lim_{\varepsilon\to 0} \eta_\varepsilon(x) = \begin{cases} (1-x^2)^{1/2}, & \text{for } |x| < 1, \\ 0, & \text{for } |x| > 1, \end{cases}$$

- For any compact subset K in the unit disk, there is $C_K > 0$ such that

$$\|\eta_\varepsilon - \eta_0\|_{C^1(K)} \leq C_K\,\varepsilon^2 .$$

- There is $C > 0$ such that

$$\|\eta_\varepsilon - \eta_0\|_{L^\infty} \leq C\,\varepsilon^{1/3}, \quad \|\nabla_x \eta_\varepsilon\|_{L^\infty} \leq C\,\varepsilon^{-1/3} .$$

Excited states (vortices) in the asymptotic theory

Let u_ε be the non-positive solution of the stationary problem (an excited state)

$$\varepsilon^2 \nabla_x^2 u_\varepsilon + (1 - |x|^2 - |u_\varepsilon|^2) u_\varepsilon = 0, \quad x \in \mathbb{R}^d.$$

If $d = 1$, u_ε is real and the excited states are classified by the number m of zeros of $u_\varepsilon(x)$ on $\mathbb{R}$.
If $d = 2$, u_ε can be complex-valued for vortex configurations (single vortices, dipoles, quadrupoles, etc).

The product representation

$$u(x,t) = \eta_\varepsilon(x) v(x,t)$$

brings the Gross–Pitaevskii equation to the equivalent form

$$i\,\varepsilon\,\eta_\varepsilon^2 v_t + \varepsilon^2 \nabla_x \left(\eta_\varepsilon^2 \nabla_x v\right) + \eta_\varepsilon^4 (1 - |v|^2) v = 0,$$

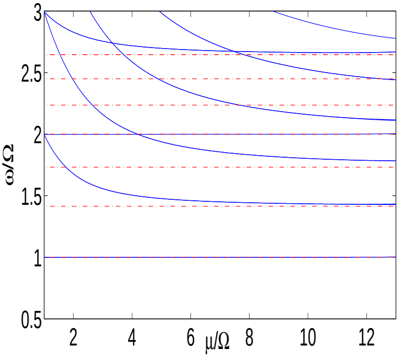
where $\lim_{|x|\to\infty} |v(x)| = 1$.

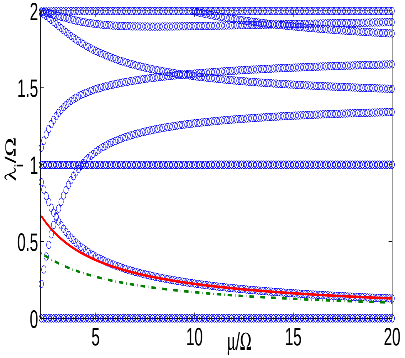
Vortices in harmonic potentials

Earlier results in physics literature:

- Möttönen *et al.* (2005) - computation of the interaction energy for two, three, and four vortices and prediction of stationary dipoles and quadrupoles
- Li *et al.* (2008) - dynamics of a vortex-antivortex pair on a phase plane
- Kevrekidis & P (2010) - numerical computations of eigenvalues of the ground state and comparison with the hydrodynamical theory of Stringari (1996)
- Kollar & Pego (2010) - numerical computations of eigenvalues for charge-one and charge-two vortices
- Middelkamp *et al.* (2010) - numerical computations of eigenvalues for single vortices, dipoles and quadrupoles.

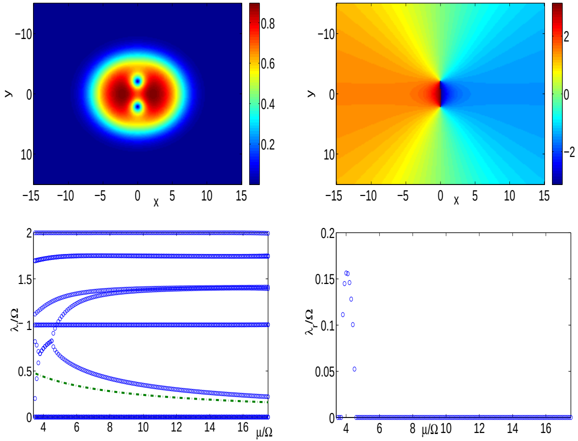
Eigenvalues of the spectral stability problem





Left: ground state η_ε. Right: charge-one vortex.

Dipole configurations



Variational construction of vortices

The equivalent Gross–Pitaevskii equation

$$i\,\varepsilon\,\eta_\varepsilon^2 v_t + \varepsilon^2\,\nabla_x\left(\eta_\varepsilon^2 \nabla_x v\right) + \eta_\varepsilon^4 (1 - |v|^2) v = 0,$$

is the Euler–Lagrange equation for the Lagrangian $L(v) = K(v) + \Lambda(v)$ with the kinetic energy

$$K(v) = \frac{i}{2}\,\varepsilon \int_{\mathbb{R}^2} \eta_\varepsilon^2 (v \bar{v}_t - \bar{v} v_t) dx$$

and the potential energy

$$\Lambda(v) = \varepsilon^2 \int_{\mathbb{R}^2} \eta_\varepsilon^2 |\nabla_x v|^2 dx + \frac{1}{2} \int_{\mathbb{R}^2} \eta_\varepsilon^4 (1 - |v|^2)^2 dx.$$

Free vortex of the defocusing NLS equation

If $\eta_\varepsilon \equiv 1$, the defocusing NLS equation has a single vortex of charge m:

$$V_m(x) = \Psi_m(R)e^{im\theta}, \quad R = r\epsilon^{-1}$$

where $m \in \mathbb{N}$ and $\Psi_m(R)$ is a solution of

$$\Psi_m'' + R^{-1}\Psi_m' - m^2R^{-2}\Psi_m + (1-\Psi_m^2)\Psi_m = 0, \quad R > 0,$$

such that $\Psi_m(0) = 0$, $\Psi_m(R) > 0$ for all $R > 0$, and $\lim_{R\to\infty} \Psi_m(R) = 1$.

The short-range asymptotics is

$$\Psi_m(R) = \alpha_m R^m + \mathcal{O}(R^{m+2}) \quad \text{as} \quad R \to 0$$

The long-range asymptotics is

$$\Psi_m^2(R) = 1 - \frac{m^2}{R^2} + \mathcal{O}\left(\frac{1}{R^4}\right) \quad \text{as} \quad R \to \infty.$$

Kinetic energy

We can use variables

$$x = x_0 + \varepsilon X, \quad y = y_0 + \varepsilon Y,$$

and write the kinetic energy as

$$K(V_m) = -\dot{x}_0 K_x(V_m) - \dot{y}_0 K_y(V_m),$$

where

$$K_x(V_m) = -m\,\varepsilon^2 \int_{\mathbb{R}^2} \eta_\varepsilon^2(x) \frac{Y \Psi_m^2}{R^2} dX dY, \quad K_y(V_m) = m\,\varepsilon^2 \int_{\mathbb{R}^2} \eta_\varepsilon^2(x) \frac{X \Psi_m^2}{R^2} dX dY.$$

Lemma

For small $\varepsilon > 0$ and small $(x_0, y_0) \in \mathbb{R}^2$, the kinetic energy of a single vortex is represented by

$$K(V_m) = \pi m\,\varepsilon (x_0 \dot{y}_0 - y_0 \dot{x}_0) \left(1 + \mathcal{O}(\varepsilon) + \mathcal{O}(x_0^2 + y_0^2)\right).$$

Justification

The symmetry of the integrand implies that $K_x(V_m)|_{y_0=0} = 0$. We can write $K_x(V_m) = J_1 + J_2$, where

$$J_1 = -m\,\varepsilon^2 \int_{\mathbb{R}^2} \eta_\varepsilon^2(x) \frac{Y(\Psi_m^2 - 1)}{R^2} dXdY, \quad J_2 = -m\,\varepsilon^2 \int_{\mathbb{R}^2} \eta_\varepsilon^2(x) \frac{Y}{R^2} dXdY.$$

For small $\varepsilon > 0$ and small $(x_0, y_0) \in \mathbb{R}^2$, there is $C > 0$ such that

$$|J_1| \leq C\,\varepsilon^2\,|y_0|, \quad |J_2| \leq C\,\varepsilon\,|y_0|.$$

Finally, we compute

$$\begin{aligned} \partial_{y_0} J_2|_{x_0=y_0=0} &= -m\,\varepsilon^2 \int_{\mathbb{R}^2} (\partial_y \eta_\varepsilon^2(r)|_{r=\varepsilon R}) \frac{Y}{R^2} dXdY \\ &= -m\,\varepsilon^2 \int_0^{2\pi} d\theta \int_0^\infty dR (\partial_r \eta_\varepsilon^2(r)|_{r=\varepsilon R}) \sin^2(\theta) \\ &= \pi m\,\varepsilon\,\eta_\varepsilon^2(0) = \pi m\,\varepsilon + \mathcal{O}(\varepsilon^3), \end{aligned}$$

Potential energy

We write the potential energy as

$$\Lambda(V_m) = \varepsilon^2 \int_{\mathbb{R}^2} \eta_\varepsilon^2(x) \left[\left(\frac{d\Psi_m}{dR} \right)^2 + \frac{m^2}{R^2} \Psi_m^2 \right] dXdY + \frac{1}{2} \varepsilon^2 \int_{\mathbb{R}^2} \eta_\varepsilon^4(x)(1-\Psi_m^2)^2 dXdY$$

Lemma

For small $\varepsilon > 0$ and small $(x_0, y_0) \in \mathbb{R}^2$, the potential energy of a single vortex is represented by

$$\Lambda(V_m) - \Lambda(V_m)|_{x_0=y_0=0} = -\pi\,\varepsilon\, m\omega_m (x_0^2 + y_0^2) \left(1 + \mathcal{O}(\varepsilon^{1/3}) + \mathcal{O}(x_0^2 + y_0^2) \right),$$

where ω_m is given by

$$\omega_m = \varepsilon\, m \left[1 - 2\log(\varepsilon) + \frac{2}{m^2} \int_0^\infty \left[\left(\frac{d\Psi_m}{dR} \right)^2 + \frac{m^2}{R^2} \left(\Psi_m^2 - \frac{R^2}{1+R^2} \right) \right] RdR \right].$$

Justification

We can write $\Lambda(V_m) = I_1 + I_2$, where

$$I_1 = \varepsilon^2 \int_{\mathbb{R}^2} \eta_\varepsilon^2(x) \left[\left(\frac{d\Psi_m}{dR} \right)^2 + \frac{m^2}{R^2} \left(\Psi_m^2 - \frac{R^2}{1+R^2} \right) \right] dXdY + \dots$$

and

$$I_2 = \varepsilon^2 m^2 \int_{\mathbb{R}^2} \frac{\eta_\varepsilon^2(x)}{1+R^2} dXdY = \varepsilon^2 m^2 \int_{\mathbb{R}^2} \frac{\eta_\varepsilon^2(x)}{\varepsilon^2 + (x-x_0)^2 + (y-y_0)^2} dxdy.$$

For small $\varepsilon > 0$ and small $(x_0, y_0) \in \mathbb{R}^2$, there is $C > 0$ such that

$$|I_1| \leq C\,\varepsilon^2, \quad |I_2| \leq C\,\varepsilon^2\,|\log(\varepsilon)|.$$

Finally, we compute

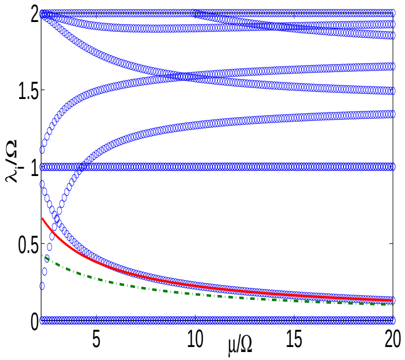
$$\begin{aligned} \partial_{x_0}^2 I_2|_{x_0=y_0=0} &= 2\,\varepsilon^2 m^2 \int_{\mathbb{R}^2} \eta_\varepsilon^2(x) \frac{3x^2 - y^2 - \varepsilon^2}{(\varepsilon^2 + x^2 + y^2)^3} dxdy \\ &= 4\pi m^2 \int_0^\infty \frac{\eta_\varepsilon^2(\varepsilon R)(R^2-1)R}{(1+R^2)^3} dR \\ &= 4\pi m^2\,\varepsilon^2 \left(\log(\varepsilon) + \frac{1}{2} \right) + \mathcal{O}(\varepsilon^{2+1/3}). \end{aligned}$$

Eigenfrequencies of the charge-one vortex

Euler–Lagrange equations for the leading part of $L(V_m) = K(V_m) + \Lambda(V_m)$ give

$$-\dot{x}_0 = \omega_m y_0, \quad \dot{y}_0 = \omega_m x_0,$$

Recall the transformation $\mu = \frac{1}{2\varepsilon}$ and $\mathrm{Im}(\lambda) = \frac{\omega}{2}$.



Free dipole

A dipole consists of a pair of the charge-one vortex and the charge-one antivortex,

$$V_d(x,y) = V_1(x - x_0, y - y_0)\bar{V}_1(x + x_0, y - y_0).$$

Note that

$$\left|\frac{\partial V_d}{\partial X}\right|^2 + \left|\frac{\partial V_d}{\partial Y}\right|^2 = \mathcal{O}(R^{-4}) \quad (1 - |V_d|^2)^2 = \mathcal{O}(R^{-4}) \quad \text{as} \quad R \to \infty.$$

Although the potential energy needs not be renormalized, the interaction energy is

$$\begin{aligned} \Lambda_R(V_d) &= \int_{\mathbb{R}^2} \left(\left|\frac{\partial V_d}{\partial X}\right|^2 + \left|\frac{\partial V_d}{\partial Y}\right|^2 + \frac{1}{2}(1 - |V_d|^2)^2 \right) dXdY \\ &= 2\pi \log(A) + \mathcal{O}(1) \quad \text{as} \quad A \to \infty, \end{aligned}$$

where $x_0 = \epsilon A$ (Ovchinnikov & Sigal, 2002).

Kinetic and potential energy

The single vortices for the stationary dipole are placed at $(x_0, 0)$ and $(-x_0, 0)$, where it will be assumed that $x_0 \to 0$ and $A = x_0/\varepsilon \to \infty$ as $\varepsilon \to 0$.

Lemma

For small $\varepsilon > 0$ and small $(x_0, y_0) \in \mathbb{R}^2$ such that x_0/ε is large as $\varepsilon \to 0$, the kinetic and potential energies of a dipole are represented by

$$K(V_d) = 2\pi m \varepsilon (x_0 \dot{y}_0 - y_0 \dot{x}_0) \left(1 + \mathcal{O}(\varepsilon) + \mathcal{O}(x_0^2 + y_0^2)\right).$$

and

$$\begin{aligned} \Lambda(V_d) - \Lambda(V_d)|_{x_0=y_0=0} &= 4\pi \varepsilon^2 (x_0^2 + y_0^2) \left(\log(\varepsilon) + \mathcal{O}(1) + \mathcal{O}(x_0^2 + y_0^2)\right) \\ &\quad + 2\pi \varepsilon^2 \left(\log(x_0/\varepsilon) + \mathcal{O}(1)\right). \end{aligned}$$

Eigenfrequencies of the dipole

Euler–Lagrange equations for the leading part of $L(V_d) = K(V_d) + \Lambda(V_d)$ give

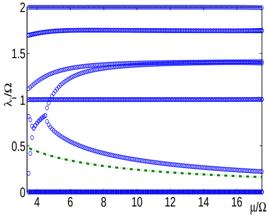
$$\left\{ \begin{array}{l} \dot{y}_0 + 2\varepsilon \log(\varepsilon) x_0 + \frac{\varepsilon}{2x_0} = 0, \\ -\dot{x}_0 + 2\varepsilon \log(\varepsilon) y_0 = 0. \end{array} \right.$$

The equilibrium state for the stationary dipole is

$$x_0 = \frac{1}{2|\log(\varepsilon)|^{1/2}}, \quad y_0 = 0,$$

and the eigenfrequency of the epicyclic precession is

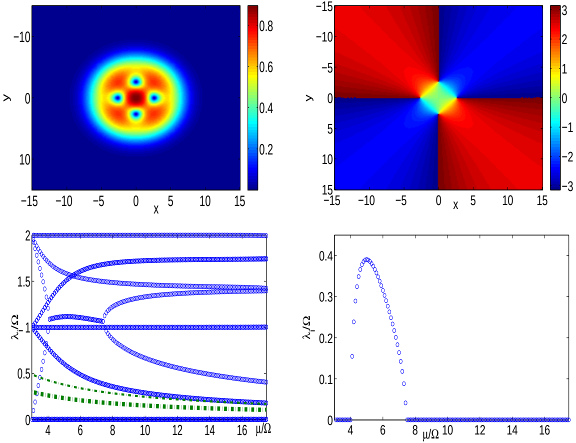
$$\omega_d = 2\sqrt{2}\,\varepsilon\,|\log(\varepsilon)| \approx \sqrt{2}\omega_1 + \mathcal{O}(\varepsilon).$$



Quadrupole

Variational ansatz

$$V_q(x,y) = V_1(x-x_0, y-y_0)\bar{V}_1(x+x_0, y-y_0)V_1(x+x_0, y+y_0)\bar{V}_1(x-x_0, y+y_0).$$



First excited state

Consider the non-positive real stationary solutions

$$\varepsilon^2 u_\varepsilon''(x) + (1 - x^2 - u_\varepsilon^2(x))u_\varepsilon(x) = 0, \quad x \in \mathbb{R}.$$

The first excited state is an odd stationary solution such that

$$u_\varepsilon(0) = 0, \quad u_\varepsilon(x) > 0 \text{ for all } x > 0, \quad \text{and} \quad \lim_{x \to \infty} u_\varepsilon(x) = 0.$$

Theorem

For sufficiently small $\varepsilon > 0$, there exists a unique solution $u_\varepsilon \in \mathcal{C}^\infty(\mathbb{R})$ with properties above and there is $C > 0$ such that

$$\left\| u_\varepsilon - \eta_\varepsilon \tanh\left(\frac{\cdot}{\sqrt{2}\,\varepsilon}\right)\right\|_{L^\infty} \leq C\,\varepsilon^{2/3}.$$

In particular, the solution converges pointwise as $\varepsilon \to 0$ to

$$u_0(x) := \lim_{\varepsilon \to 0} u_\varepsilon(x) = \eta_0(x)\,\mathrm{sign}(x), \quad x \in \mathbb{R}.$$

Steps of the proof

Step 1: Decomposition.
We substitute

$$u_\varepsilon(x) = \eta_\varepsilon(x) \tanh\left(\frac{x}{\sqrt{2}\,\varepsilon}\right) + w_\varepsilon(x)$$

and obtain

$$L_\varepsilon w_\varepsilon = H_\varepsilon + N_\varepsilon(w_\varepsilon),$$

where

$$L_\varepsilon := -\varepsilon^2 \partial_x^2 + x^2 - 1 + 3\eta_\varepsilon^2(x) \tanh^2\left(\frac{x}{\sqrt{2}\,\varepsilon}\right),$$

$$H_\varepsilon(x) := \eta_\varepsilon(x)\left(\eta_\varepsilon^2(x) - 1\right) \mathrm{sech}^2\left(\frac{x}{\sqrt{2}\,\varepsilon}\right) \tanh\left(\frac{x}{\sqrt{2}\,\varepsilon}\right) + \sqrt{2}\,\varepsilon\,\eta_\varepsilon'(x) \mathrm{sech}^2\left(\frac{x}{\sqrt{2}\,\varepsilon}\right)$$

and

$$N_\varepsilon(w_\varepsilon)(x) = -3\eta_\varepsilon(x) \tanh\left(\frac{x}{\sqrt{2}\,\varepsilon}\right) w_\varepsilon^2(x) - w_\varepsilon^3(x).$$

Steps of the proof

Step 2: Linear estimates.
Using variable $x = \sqrt{2}\,\varepsilon\, z$, we obtain

$$\hat{L}_\varepsilon = -\frac{1}{2}\partial_z^2 + 2\,\varepsilon^2 z^2 - 1 + 3\hat{\eta}_\varepsilon^2(z)\tanh^2(z) = \hat{L}_0 + \hat{U}_\varepsilon(z),$$

where

$$\hat{L}_0 := -\frac{1}{2}\partial_z^2 + 2 - 3\mathrm{sech}^2(z)$$

and

$$\hat{U}_\varepsilon(z) := 2\,\varepsilon^2 z^2 + 3(\hat{\eta}_\varepsilon^2(z) - 1)\tanh^2(z).$$

The spectrum of $\hat{L}_0$ consists of two eigenvalues at 0 and $\frac{3}{2}$ with eigenfunctions $\mathrm{sech}^2(z)$ and $\tanh(z)\mathrm{sech}(z)$ and the continuous spectrum on $[2, \infty)$.

Steps of the proof

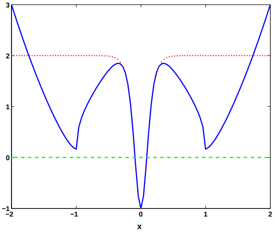


Figure: Potentials of operators L_ε (solid line) and L_0 (dots) for the first excited state.

Resolvent of the unperturbed operator:

$$\exists C > 0,\ \alpha > 0 : \quad \forall \hat{f} \in L^2_{\rm odd}(\mathbb{R}) \cap L^\infty_\alpha(\mathbb{R}) : \quad \|\hat{L}_0^{-1}\hat{f}\|_{H^2 \cap L^\infty_\alpha} \leq C \|\hat{f}\|_{L^2 \cap L^\infty_\alpha}.$$

Resolvent of the full operator:

$$\exists C > 0 : \quad \forall \hat{f} \in L^2_{\rm odd}(\mathbb{R}) : \quad \|\hat{L}_\varepsilon^{-1}\hat{f}\|_{H^2} \leq C\,\varepsilon^{-2/3}\,\|\hat{f}\|_{L^2}.$$

Steps of the proof

Step 3: Bounds on the inhomogeneous and nonlinear terms.
Recall that we are solving

$$L_\varepsilon w_\varepsilon = H_\varepsilon + N_\varepsilon(w_\varepsilon),$$

where

$$\hat{H}_\varepsilon \in L^2_{\rm odd}(\mathbb{R}) \quad \text{and} \quad \hat{N}_\varepsilon(\hat{w}_\varepsilon) : H^2_{\rm odd}(\mathbb{R}) \mapsto L^2_{\rm odd}(\mathbb{R}).$$

For any $\varepsilon > 0$ and $\alpha \in (0,2)$, we have

$$\begin{aligned} \|\hat{H}_\varepsilon\|_{L^2 \cap L^\infty_\alpha} &\leq \|\eta_\varepsilon\|_{L^\infty} \|(1-\hat{\eta}_\varepsilon^2){\rm sech}^2(\cdot)\|_{L^2 \cap L^\infty_\alpha} + \sqrt{2}\,\varepsilon\,\|\eta'_\varepsilon\|_{L^\infty} \|{\rm sech}^2(\cdot)\|_{L^2 \cap L^\infty_\alpha} \\ &\leq C\,\varepsilon^{2/3}. \end{aligned}$$

For any $\hat{w}_\varepsilon \in H^2(\mathbb{R})$, we have

$$\|\hat{N}_\varepsilon(\hat{w}_\varepsilon)\|_{L^2} \leq 3\|\eta_\varepsilon\|_{L^\infty}\|\hat{w}_\varepsilon^2\|_{H^2} + \|\hat{w}_\varepsilon^3\|_{H^2} \leq 3\|\hat{w}_\varepsilon\|^2_{H^2} + \|\hat{w}_\varepsilon\|^3_{H^2}.$$

Steps of the proof

Step 4: Normal-form transformation.
Let

$$\hat{w}_\varepsilon = \hat{w}_1 + \hat{w}_2 + \hat{\varphi}_\varepsilon, \quad \hat{w}_1 = \hat{L}_0^{-1}\hat{H}_\varepsilon, \quad \hat{w}_2 = -3\hat{L}_0^{-1}\hat{\eta}_\varepsilon \tanh(z)\hat{w}_1^2,$$

where

$$\exists C > 0 : \quad \|\hat{w}_1\|_{H^2\cap L^\infty_\alpha} \leq C\,\varepsilon^{2/3}, \quad \|\hat{w}_2\|_{H^2\cap L^\infty_\alpha} \leq C\,\varepsilon^{4/3}.$$

The remainder term $\hat{\varphi}_\varepsilon$ solves the new problem

$$\mathcal{L}_\varepsilon\hat{\varphi}_\varepsilon = \mathcal{H}_\varepsilon + \mathcal{N}_\varepsilon(\hat{\varphi}_\varepsilon),$$

where

$$\|\mathcal{H}_\varepsilon\|_{L^2} \leq C\,\varepsilon^2,$$

$$\forall\hat{\varphi}_\varepsilon \in B_\delta(H^2_{\rm odd}) : \quad \|\mathcal{N}_\varepsilon(\hat{\varphi}_\varepsilon)\|_{L^2} \leq C(\delta)\|\hat{\varphi}_\varepsilon\|^2_{H^2},$$

and

$$\forall\hat{\varphi}_\varepsilon, \hat{\phi}_\varepsilon \in B_\delta(H^2_{\rm odd}) : \quad \|\mathcal{N}_\varepsilon(\hat{\varphi}_\varepsilon) - \mathcal{N}_\varepsilon(\hat{\phi}_\varepsilon)\|_{L^2} \leq C(\delta)\left(\|\hat{\varphi}_\varepsilon\|_{H^2} + \|\hat{\phi}_\varepsilon\|_{H^2}\right)\|\hat{\varphi}_\varepsilon - \hat{\phi}\|_{H^2}.$$

Steps of the proof

Step 5: Fixed-point arguments.
Since

$$\exists C > 0 : \quad \forall \hat{f} \in L^2_{\rm odd}(\mathbb{R}) : \quad \|\mathcal{L}_\varepsilon^{-1} \hat{f}\|_{H^2} \leq C\, \varepsilon^{-2/3}\, \|\hat{f}\|_{L^2},$$

the map $\hat{\varphi}_\varepsilon \mapsto \mathcal{L}_\varepsilon^{-1} \mathcal{N}_\varepsilon(\hat{\varphi}_\varepsilon)$ is a contraction in the ball $B_\delta(H^2_{\rm odd})$ if $\delta \ll \varepsilon^{2/3}$.

On the other hand, the source term $\mathcal{L}_\varepsilon^{-1} \mathcal{H}_\varepsilon$ is as small as $\mathcal{O}(\varepsilon^{4/3})$. Therefore, Banach's Fixed-Point Theorem applies in the ball $B_\delta(H^2_{\rm odd})$ with $\delta \sim \varepsilon^{4/3}$.

Step 6: Properties of $u_\varepsilon(x)$. It remains to prove that $u_\varepsilon(x) > 0$ for all $x > 0$. This property does not come immediately from the fixed-point solution

$$u_\varepsilon(x) = \eta_\varepsilon(x) \tanh\left(\frac{x}{\sqrt{2}\,\varepsilon}\right) + w_\varepsilon(x),$$

where $\|w_\varepsilon\|_{L^\infty} \leq C\, \varepsilon^{2/3}$.

Second excited state

The second excited state is an odd stationary solution such that

$$u_\varepsilon(x) > 0 \ \text{ for all } \ |x| > x_0, \quad u_\varepsilon(x) < 0 \ \text{ for all } \ |x| < x_0, \quad \text{and} \quad \lim_{x\to\infty} u_\varepsilon(x) = 0.$$

Theorem

For sufficiently small $\varepsilon > 0$, there exists a unique solution $u_\varepsilon \in \mathcal{C}^\infty(\mathbb{R})$ with properties above and there exist $a > 0$ and $C > 0$ such that

$$\left\| u_\varepsilon - \eta_\varepsilon \tanh\left(\frac{\cdot - a}{\sqrt{2}\,\varepsilon}\right) \tanh\left(\frac{\cdot + a}{\sqrt{2}\,\varepsilon}\right) \right\|_{L^\infty} \leq C\,\varepsilon^{2/3}$$

and

$$a = -\frac{\varepsilon}{\sqrt{2}} \left(\log(\varepsilon) + \frac{1}{2} \log|\log(\varepsilon)| - \frac{3}{2}\log(2) + o(1) \right) \quad \text{as} \quad \varepsilon \to 0.$$

In particular, $x_0 = a + \mathcal{O}(\varepsilon^{5/3})$.

Steps of the proof

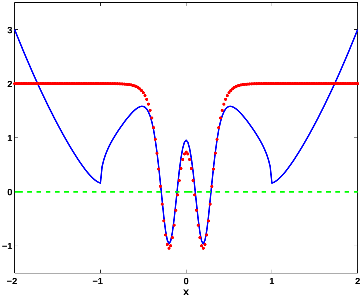


Figure: Potential of operator L_ε (solid line) and L_0 (dots) for the second excited state.

Here the leading-order operator

$$\hat{L}_0(\zeta) = -\frac{1}{2}\partial_z^2 + 2 - 3\text{sech}^2(z+\zeta) - 3\text{sech}^2(z-\zeta), \quad \zeta = \frac{a}{\sqrt{2}\,\varepsilon},$$

has two eigenvalues in the neighborhood of 0 for large ζ because of the double-well potential centered at $z = \pm\zeta$.

Summary of our results

- We justified asymptotic representations of the ground and excited states
- We predicted asymptotic dependence of the distance between individual solitons/vortices for m-excited states.
- We predicted asymptotic dependence of the eigenfrequencies of oscillations for m-excited states related to the dynamics of solitons/vortices with respect to each other and to the harmonic potential.
- We illustrated both asymptotic predictions numerically.