Eigenvalues of nonlinear bound states in the Thomas–Fermi approximation

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Introduction

Density waves in cigar–shaped Bose–Einstein condensates are modeled by the Gross-Pitaevskii equation

$$iu_t + \varepsilon^2 u_{xx} + (1 - x^2)u - |u|^2 u = 0,$$

where ε is a small parameter.

Limit $\varepsilon \to 0$ is referred to as the hydrodynamics limit or as the Thomas–Fermi approximation since the work of L.H. Thomas (1927) and E. Fermi (1928).

Theorem(Brezis-Oswald, 1986): There exists a real-valued, positive-definite global minimizer of the Gross–Pitaevskii energy

$$E_\varepsilon(u) = \int_{\mathbb{R}} \left(\frac{1}{2} \varepsilon^2 |u_x|^2 + \frac{1}{2}(x^2 - 1)|u|^2 + \frac{1}{4}|u|^4 \right) dx$$

in the energy space

$$\mathcal{H}_1 = \left\{ u \in H^1(\mathbb{R}) : \quad xu \in L^2(\mathbb{R}) \right\},$$

for sufficiently small $\varepsilon > 0$.

Ground state of energy

Let η_ε be a global minimizer of E_ε. From Euler–Lagrange equations, it solves

$$-\varepsilon^2 \eta_\varepsilon''(x) + \left(\eta_\varepsilon^2 + x^2 - 1\right) \eta_\varepsilon = 0, \quad \forall x \in \mathbb{R}.$$

The formal limit for the ground state is

$$\eta_0(x) = \begin{cases} (1 - x^2)^{1/2}, & \text{for } |x| < 1, \\ 0, & \text{for } |x| > 1, \end{cases}$$

Recently, Aftalion,Alama, & Bronsard (2005) and Ignat & Millot (2006) justified convergence to the Thomas-Fermi approximation and proved

$$\begin{cases} (1 - C\,\varepsilon^{1/3}) \leq \frac{\eta_\varepsilon(x)}{(1-x^2)^{1/2}} \leq 1 & \text{for } |x| \leq 1 - \varepsilon^{2/3} \\ 0 \leq \eta_\varepsilon(x) \leq C\,\varepsilon^{1/3} \exp\left(\frac{1-x^2}{4\,\varepsilon^{2/3}}\right) & \text{for } |x| \geq 1, \end{cases}$$

for some $C > 0$ uniformly in $0 < \varepsilon \ll 1$.

Spectral stability

Linearization of the Gross–Pitaevskii equation with

$$u(x,t) = \eta_\varepsilon(x) + [u(x) + iw(x)]\, e^{\lambda t} + [\bar{u}(x) - i\bar{w}(x)]\, e^{\bar{\lambda} t} + \mathcal{O}(\|u\|^2 + \|w\|^2)$$

results in the non-self-adjoint eigenvalue problem

$$\left\{ \begin{array}{rcr} -\varepsilon^2 u'' + (x^2 - 1 + 3\eta_\varepsilon^2)u & = & -\lambda w, \\ -\varepsilon^2 w'' + (x^2 - 1 + \eta_\varepsilon^2)w & = & \lambda u, \end{array} \right.$$

or, equivalently, in the generalized eigenvalue problem

$$\left(-\varepsilon^2 \partial_x^2 + x^2 - 1 + \eta_\varepsilon^2\right) w = \gamma \left(-\varepsilon^2 \partial_x^2 + x^2 - 1 + 3\eta_\varepsilon^2\right)^{-1} w,$$

where $\gamma = -\lambda^2$.

We are concerned here with eigenvalues of the spectral problem in the limit $\varepsilon \to 0$. In the present time, we have results when η_ε is replaced by η_0. Results for $\eta_\varepsilon = \eta_0 + \mathcal{O}_{L^\infty}(\varepsilon^{1/3})$ will require more work.

Eigenvalues in the hydrodynamics limit

Consider the generalized eigenvalue problem

$$\left(-\varepsilon^2\,\partial_x^2 + x^2 - 1 + \eta_0^2\right) w = \gamma\left(-\varepsilon^2\,\partial_x^2 + x^2 - 1 + 3\eta_0^2\right)^{-1} w$$

and restrict it on $(-1, 1)$ as

$$-\left(-\varepsilon^2\,\partial_x^2 + 2(1 - x^2)\right) w''(x) = \gamma\,\varepsilon^{-2}\,w(x).$$

Let $\Gamma = \gamma\,\varepsilon^{-2}$ and drop $\varepsilon^2\,\partial_x^2$ term to obtain the singular Sturm–Liouville problem

$$-2(1 - x^2)w''(x) = \Gamma w(x), \quad -1 < x < 1.$$

Lemma: The only $\mathcal{C}^2$ solutions on $[-1, 1]$ with $w(1) = w(-1) = 0$ are Gegenbauer polynomials $w = C_{n+1}^{-1/2}(x)$ for $\Gamma = \Gamma_n := 2n(n+1)$, $n \geq 1$.

Stringari (1996); Fliesser et al. (1997); Eberlein et al. (2005); and others

Main results

Theorem: Linearized problem for sufficiently small $\varepsilon > 0$ has a purely discrete spectrum that consists of eigenvalues at $\{\Gamma_n^\varepsilon\}_{n\in\mathbb{N}}$ sorted in the increasing order and

$$\Gamma_n^\varepsilon \longrightarrow \Gamma_n \quad \text{as } \varepsilon \to 0$$

for every fixed $n \in \mathbb{N}$.

Claim: There exists $C_n > 0$ such that

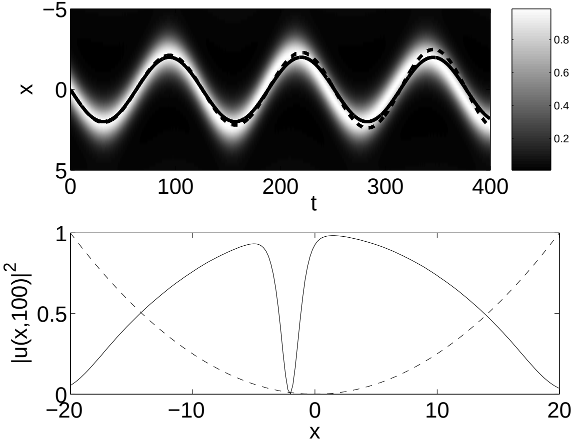
$$|\Gamma_n^\varepsilon - \Gamma_n| \leq C_n \, \varepsilon^{1/3}$$

for sufficiently small $\varepsilon > 0$.

Remark: The convergence rate of eigenvalues may not be sharp and numerical results indicate that the convergence rate is $\mathcal{O}(\varepsilon^2)$ for a fixed $n \in \mathbb{N}$.

Possible applications

Oscillations of 1-dim vortices:



D.P. & P. Kevrekidis, Cont.Math. (2008)
D.P. & P. Kevrekidis, ZAMP (2008)

1-dim vortex in the hydrodynamics limit

Gross–Pitaevskii equation

$$iU_\tau + U_{\xi\xi} + (\mu - \xi^2)U - |U|^2 U = 0$$

reduces to

$$iu_t + \varepsilon^2 u_{xx} + (1 - x^2)u - |u|^2 u = 0,$$

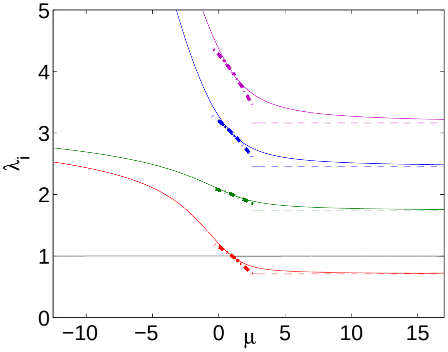
as $\mu \to \infty$ by rescaling

$$x = \varepsilon^{1/2}\xi, \quad t = \varepsilon^{-1}\tau, \quad u(x,t) = \varepsilon^{1/2}U(\xi,\tau), \quad \mu = \varepsilon^{-1}.$$

1-dim vortex is a solution in the form $v_\varepsilon(x)\eta_\varepsilon(x)$, where $v_\varepsilon(-x) = -v_\varepsilon(x)$ with a single zero at $x = 0$. Hydrodynamics limit of the 1-dim vortex is

$$v_0(x) = \mathrm{sign}(x).$$

Eigenvalues of the 1-dim vortex



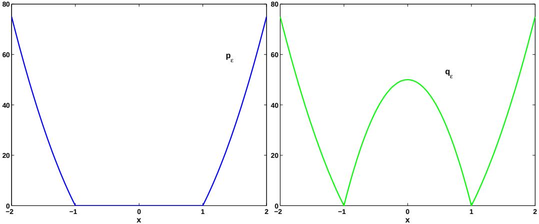
Limiting values as $\mu \to \infty$ correspond to eigenvalues of the ground state $\{\Gamma_n\}_{n\in\mathbb{N}}$ plus an additional (smallest) eigenvalue for the 1-dim vortex.

Compact operators of the linearized problem

Eigenvalue problem can be formulated as $A_\varepsilon w = \mu w$, where $\mu = \Gamma^{-1}$ and

$$A_\varepsilon := \varepsilon^{-2}(-\partial_x^2 + p_\varepsilon(x))^{-1}(-\partial_x^2 + q_\varepsilon(x))^{-1} = \varepsilon^{-2}(L_-^\varepsilon)^{-1}(L_+^\varepsilon)^{-1},$$

$$p_\varepsilon(x) = \varepsilon^{-2}(x^2-1)\mathbf{1}_{\{|x|>1\}}, \quad q_\varepsilon(x) = \varepsilon^{-2}\left[2(1-x^2)\mathbf{1}_{\{|x|<1\}} + (x^2-1)\mathbf{1}_{\{|x|>1\}}\right].$$



Both $L_\pm^\varepsilon$ are positive, self-adjoint, and invertible operators with a compact resolvent. Therefore, A_ε is a compact operator on $L^2(\mathbb{R})$ for any fixed $\varepsilon > 0$. Moreover, eigenvalues are strictly positive since A_ε is self-similar to $(L_+^\varepsilon)^{-1/2}(L_-^\varepsilon)^{-1}(L_+^\varepsilon)^{-1/2}$.

Limiting operator

As $\varepsilon \to 0$, we can formally expect that A_ε converges in some sense to

$$A_0 = (-\partial_x^2 + p_0)^{-1} \frac{1}{2(1-x^2)}, \quad p_0(x) = \begin{cases} 0 & \text{if } |x| < 1, \\ +\infty & \text{if } |x| > 1. \end{cases}$$

Properties of A_0:

- For any $u \in L^2(\mathbb{R})$, $A_0 u \in L^2(\mathbb{R})$ and

$$\begin{cases} (A_0 u)_{|\{|x|>1\}} \equiv 0, \\ (A_0 u)_{|(-1,1)} = (-\Delta_D)^{-1} \left(\frac{u}{2(1-x^2)} \right)_{|(-1,1)}. \end{cases}$$

  where Δ_D is the Dirichlet realization of ∂_x^2 on $[-1, 1]$.
- $A_0 u$ is continuous on $\mathbb{R}$ so that $(A_0 u)(\pm 1) = 0$
- A_0 is compact on $L^2(\mathbb{R})$.

Spectrum of A_0

The spectrum of A_0 is purely discrete. 0 is an eigenvalue with an infinite-dimensional subspace of eigenfunctions with a support on $\{x \in \mathbb{R} : \quad |x| > 1\}$. Non-zero eigenvalues are found from

$$-2(1-x^2)w''(x) = \mu^{-1}w(x), \quad -1 < x < 1,$$

subject to $w(\pm 1) = 0$. Let $z = x^2$, $u(z) = w(x)$, and write it as the hypergeometric equation

$$z(1-z)u''(z) + \frac{1}{2}(1-z)u'(z) + \frac{1}{8\mu}u(z) = 0, \quad 0 < z < 1.$$

The only solutions with $u(1) = 0$ are polynomials for $\mu = \mu_n = \frac{1}{2n(n+1)}$ for an integer $n \geq 1$. Therefore,

$$\sigma(A_0) = \left\{ \frac{1}{2n(n+1)}, \ \ n \geq 1 \right\} \cup \{0\}.$$

On the proof of the main theorem

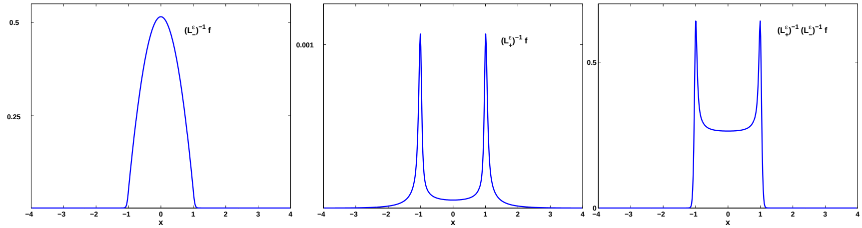
Main Theorem follows from the claim that $A_\varepsilon \to A_0$ as $\varepsilon \to 0$ in the L^2 norm, that is

$$\forall u, \phi \in L^2(\mathbb{R}) : \quad \langle A_0 u - A_\varepsilon u, \phi \rangle_{L^2, L^2} \leq C(\varepsilon) \|u\|_{L^2} \|\phi\|_{L^2}$$

and $C(\varepsilon) \to 0$ as $\varepsilon \to 0$.

The idea of the proof:

- $\|(L_-^\varepsilon)^{-1} f\|_{L^\infty(|x|>1)} \lesssim \varepsilon^{2/3} \|f\|_{L^2(\mathbb{R})}$.
- $\|(L_+^\varepsilon)^{-1} f\|_{L^\infty(\mathbb{R})} \lesssim \varepsilon \|f\|_{L^2(\mathbb{R})}$.
- $\|\varepsilon^{-2} (L_+^\varepsilon)^{-1} (L_-^\varepsilon)^{-1} f\|_{L^\infty(\mathbb{R})} \lesssim \varepsilon^{-\delta} \|f\|_{L^2(\mathbb{R})}$ for a small $\delta > 0$.



ODE system for eigenfunctions

Let w be an eigenvector of A_ε for eigenvalue $\mu = \gamma^{-1}$. It solves formally the outer problem

$$\varepsilon^2 \left(-\partial_x^2 + \varepsilon^{-2}(x^2-1)\right)^2 w(x) = \gamma w(x), \quad \text{for} \quad |x| > 1$$

and the inner problem

$$-2(1-x^2)w''(x) + \varepsilon^2 w''''(x) = \gamma w(x), \quad \text{for} \quad |x| < 1.$$

Because $(L_+^\varepsilon)^{-1} w \in H^2(\mathbb{R}) \subset \mathcal{C}^1(\mathbb{R})$, $w(x)$ is $\mathcal{C}^2(\mathbb{R})$ with jump discontinuities at $x = \pm 1$:

$$w'''|_{x=1-0}^{x=1+0} = \frac{2}{\varepsilon^2} w(1), \qquad w'''|_{x=-1+0}^{x=-1-0} = \frac{2}{\varepsilon^2} w(-1)$$

For simplicity, we can look for even eigenfunctions $w(-x) = w(x)$.

Solution on the outer interval

Let $U(a;z) \equiv D_{-a-1/2}(z)$ be the Whittaker function of the parabolic cylinder equation

$$u''(z) = \left(a + \frac{z^2}{4}\right) u(z).$$

Then,

$$w(x) = c_+ U(a_+; z) + c_- U(a_-; z), \qquad x > 1$$

where

$$z = \frac{\sqrt{2}x}{\sqrt{\varepsilon}}, \qquad a_\pm = \frac{-1 \pm \varepsilon\sqrt{\gamma}}{2\varepsilon}.$$

Near $x = 1$, $U(a;z)$ is expanded asymptotically via Airy function, which gives

$$\lim_{\varepsilon\to 0} \frac{w_\varepsilon(1)}{\varepsilon^{2/3} w_\varepsilon'(1)} = \lim_{\varepsilon\to 0} \frac{w_\varepsilon''(1)}{\varepsilon^{2/3} w_\varepsilon'''(1^-)} = \frac{\mathrm{Ai}(0)}{2^{1/3}\mathrm{Ai}'(0)}.$$

Solution on the inner interval

Remark: For eigenvalue problem $L_-^\varepsilon w = \lambda w$, we have an analytic solution

$$w = \begin{cases} \cos(\sqrt{\lambda} x) & \text{for } |x| < 1, \\ cU(a; z) & \text{for } |x| > 1, \end{cases}$$

where c is constant. Then, if λ_n is the root of $\cos(\sqrt{\lambda}) = 0$, then $|\lambda_n^\varepsilon - \lambda_n| \leq C_n \, \varepsilon^{2/3}$ for a fixed $n \in \mathbb{N}$ and the bound is sharp.

Unfortunately, no explicit solutions are available for the problem $L_-^\varepsilon w = \gamma (L_+^\varepsilon)^{-1} w$ on $[-1, 1]$. So, we shall approximate solutions numerically.

Let us consider even eigenfunctions $w(x)$ in x.

Solution on the inner interval

Define two particular solutions of the system by boundary conditions

$$\left\{ \begin{array}{llll} w_1(1) = 1, & w_1''(1) = 0, & w_1'(0) = 0, & w_1'''(0) = 0, \\ w_2(1) = 0, & w_2''(1) = 1, & w_2'(0) = 0, & w_2'''(0) = 0. \end{array} \right.$$

Then,

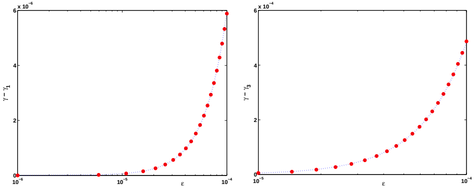
$$w(x) = a_1 w_1(x) + a_2 w_2(x), \quad 0 < x < 1$$

and the matching conditions at $x = 1$ set up a linear homogeneous system on (a_1, a_2, c_+, c_-), which has nonzero solutions if $D(\gamma; \varepsilon) = 0$, where $D(\gamma; \varepsilon)$ is analytic in $\gamma > 0$ and $\varepsilon > 0$.

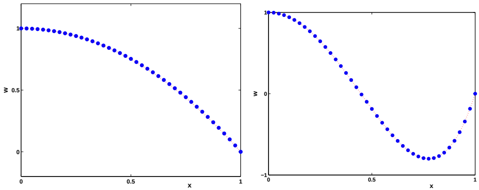
Simple zeros of $D(\gamma; \varepsilon)$ are structurally stable and can be traced as $\varepsilon \to 0$. We investigate two smallest zero near $\gamma_1 = 4$ and $\gamma_3 = 24$.

Numerical results

Rate of convergence:



Even eigenfunctions:



Numerical convergence rate suggests $|\gamma_n^{\varepsilon} - \gamma_n| \leq C_n \, \varepsilon^2$.

Further problems

- Prove the claim $|\gamma_n^\varepsilon - \gamma_n| \leq C_n \, \varepsilon^{1/3}$ rigorously.
- Extend the bound to justify the numerical convergence rate of $\mathcal{O}(\varepsilon^2)$.
- Generalize the analysis to nonlinear ground states $\eta_\varepsilon = \eta_0 + \mathcal{O}_{L^\infty}(\varepsilon^{1/3})$.
- Consider eigenvalues of the 1-dim vortices.