Translationally invariant NLS lattices

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References:

D.P., V. Rothos, Physica D 202, 16-36 (2005)

O. Oxtoby, D.P., I. Barashenkov, Nonlinearity 19, 217-235 (2006)

G. Iooss, D.P., Physica D 216, 327 (2006)

D.P., Nonlinearity, accepted (2006)

SIAM Conference on Nonlinear Waves, September 10-13, 2006

Discrete nonlinear Schrödinger model

Continuous NLS model

$$iu_t = u_{xx} + |u|^2 u, \quad x \in \mathbb{R}, \ \ u \in \mathbb{C}$$

admits traveling pulse solutions

$$u(x,t) = \sqrt{\omega}\,\mathrm{sech}(\sqrt{\omega}(x - 2ct - s))\, e^{ic(x-ct)+i\omega t+i\theta},$$

where $\omega \in \mathbb{R}_+$ and $(c, s, \theta) \in \mathbb{R}^3$.

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"Standard" (on-site) discretisation

$$i\dot{u}_n = \frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} + |u_n|^2 u_n, \quad n \in \mathbb{Z}$$

does not have "true" traveling pulse solutions.

Reductions for traveling waves

Traveling waves

$$
\begin{aligned}
u_1(t) &= u_0(t-\tau)e^{i\theta}, \\
u_2(t) &= u_1(t-\tau)e^{i\theta} = u_0(t-2\tau)e^{2i\theta}, \\
&\ldots \\
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Traveling solutions

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u_n(t) = \phi(z)e^{i\omega t}, \quad z = hn - ct, \quad c = h/\tau, \quad \omega = c\theta/h.
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Traveling solutions

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The differential advanced-delay equation

$$ic\phi'(z) = \frac{\phi(z+h) - 2\phi(z) + \phi(z-h)}{h^2} - \omega\phi(z) + |\phi|^2\phi$$

Obstacles on existence

Classical solutions $\phi(z)$ on $z \in \mathbb{R}$

- $\phi(z)$ is $C^0(\mathbb{R})$ if $c = 0$
- $\phi(z)$ is $C^1(\mathbb{R})$ if $c \neq 0$
- $\phi(z)$ is a single-humped solution decaying to $\phi = 0$

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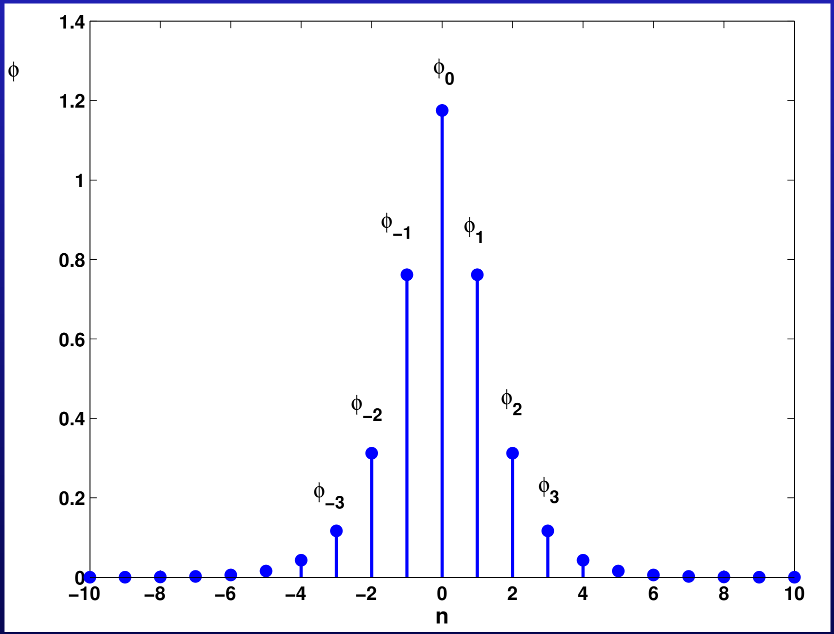
Properties of "standard" stationary solutions ($c = 0$):

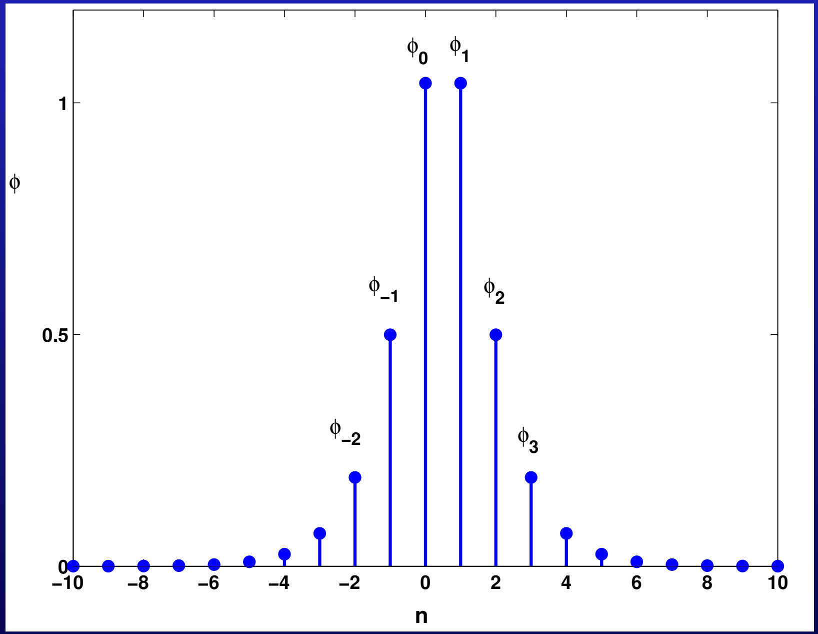
- $\phi(z)$ is piecewise constant on $z \in \mathbb{R}$
- $\phi_n = \phi(nh)$ is symmetric either about a node or about the midpoint between two nodes
- No continuous deformation exists between these two particular solutions (Peierls–Nabarro potential)

Example of stationary solutions

Stationary solutions in the "standard" discrete NLS model

$$\frac{\phi_{n+1} - 2\phi_n + \phi_{n-1}}{h^2} - \phi_n + \phi_n^3 = 0, \quad n \in \mathbb{Z}$$





Exceptional discretizations

General discrete NLS equation:

$$i\dot{u}_n = \frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} + f(u_{n-1}, u_n, u_{n+1})$$

where

- P1 (continuity) $f(u,u,u) = 2|u|^2 u$
- P2 (symmetry) $f(v,u,w) = f(w,u,v)$
- P3 (gauge) $f(e^{i\alpha}v, e^{i\alpha}u, e^{i\alpha}w) = e^{i\alpha} f(v,u,w)\ \forall \alpha \in \mathbb{R}$
- P4 $f(v,u,w)$ is independent on h
- P5 $f(v,u,w)$ is homogeneous cubic polynomial in (v,u,w)

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Exceptional nonlinearities are those that support continuous stationary solutions with $c = 0$ and $\phi \in C^0(\mathbb{R})$

Examples of exceptional discretizations

Ablowitz–Ladik lattice:

$$f = (u_{n+1} + u_{n-1})\,|u_n|^2$$

New 2-parameter lattice:

$$\begin{aligned} f = (1-\chi-2\eta)|u_n|^2(u_{n+1}+u_{n-1}) + \chi u_n^2(\bar{u}_{n+1}+\bar{u}_{n-1}) \\ +\eta(|u_{n+1}|^2+|u_{n-1}|^2)(u_{n+1}+u_{n-1}) \end{aligned}$$

Cases $(\chi,\eta) = (\frac{1}{2},0)$ and $(\chi,\eta) = (0,\frac{1}{2})$ are reported in
S. Dmitriev, P. Kevrekidis, A. Sukhorukov, et al.,
Phys. Lett. A 356, 324 (2006)

Purposes of this work

- Find the most general exceptional nonlinearity from the reduction of the second-order difference equation to the first-order difference equation.

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- Confirm that this reduction for *stationary* solutions is equivalent to conservation of momentum for *time-dependent* solutions (Kevrekidis, 2003), where the momentum is

$$M = i \sum_{n \in \mathbb{Z}} \left(\bar{u}_{n+1} u_n - u_{n+1} \bar{u}_n \right).$$

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$$M = i \sum_{n \in \mathbb{Z}} \left(\bar{u}_{n+1} u_n - u_{n+1} \bar{u}_n \right).$$

- Prove that this reduction gives a *sufficient* condition for existence of translationally invariant stationary solutions.
- Apply the normal form reduction (P, Rothos, 2005) as a *necessary* condition for existence of traveling solutions.

Reductions of difference equations

Consider the second-order difference equation

$$\frac{\phi_{n+1} - 2\phi_n + \phi_{n-1}}{h^2} - \omega\phi_n + f(\phi_{n-1}, \phi_n, \phi_{n+1}) = 0$$

and reduce the problem to the first-order difference equation

$$E_n = \frac{1}{h^2}|\phi_{n+1} - \phi_n|^2 - \frac{1}{2}\omega(\phi_n\bar{\phi}_{n+1} + \bar{\phi}_n\phi_{n+1}) + g(\phi_n, \phi_{n+1}) = E_0,$$

where

P1 (continuity) $g(u,u) = |u|^4$ P2 (symmetry) $g(u,w) = g(w,u)$

P3 (gauge) $g(e^{i\alpha}u, e^{i\alpha}w) = g(u,w)\ \forall\alpha \in \mathbb{R}$

P4 $g(u,w)$ is independent on h

P5 $g(u,w)$ is homogeneous quartic polynomial in (u,w)

Constraints on the polynomial functions

The cubic polynomial f:

$$
\begin{aligned}
f \;&=\; \alpha_1|u_n|^2u_n + \alpha_2|u_n|^2(u_{n+1}+u_{n-1}) + \alpha_3 u_n^2(\bar{u}_{n+1}+\bar{u}_{n-1})\\
&+\; \alpha_4(|u_{n+1}|^2+|u_{n-1}|^2)u_n + \alpha_5(\bar{u}_{n+1}u_{n-1}+u_{n+1}\bar{u}_{n-1})u_n\\
&+\; \alpha_6(u_{n+1}^2+u_{n-1}^2)\bar{u}_n + \alpha_7 u_{n+1}u_{n-1}\bar{u}_n + \alpha_8(|u_{n+1}|^2u_{n+1} + |u_{n-1}\\
&+\; \alpha_9(u_{n+1}^2\bar{u}_{n-1}+\bar{u}_{n+1}u_{n-1}^2) + \alpha_{10}(|u_{n+1}|^2u_{n-1}+|u_{n-1}|^2u_{n+1}),
\end{aligned}
$$

The quartic polynomial g:

$$
\begin{aligned}
g = \gamma_1(|\phi_n|^2+|\phi_{n+1}|^2)(\bar{\phi}_{n+1}\phi_n+\phi_{n+1}\bar{\phi}_n) + \gamma_2|\phi_n|^2|\phi_{n+1}|^2\\
+\gamma_3(\phi_n^2\bar{\phi}_{n+1}^2+\bar{\phi}_n^2\phi_{n+1}^2) + \gamma_4(|\phi_n|^4+|\phi_{n+1}|^4),
\end{aligned}
$$

The constraints for existence of reduction:

$$
\alpha_4=\alpha_1-\alpha_6, \quad \alpha_5=\alpha_6, \quad \alpha_7=\alpha_1-2\alpha_6, \quad \alpha_{10}=\alpha_8-\alpha_9
$$

Remarks on conserved quantities

- These constraints are *equivalent* to the conditions for conservation of the momentum M:

$$M = i \sum_{n \in \mathbb{Z}} \left(\bar{u}_{n+1} u_n - u_{n+1} \bar{u}_n \right).$$

- These constraints are *incompatible* with the conditions for existence of the Hamiltonian structure:

$$i \dot{u}_n = \frac{\partial H}{\partial \bar{u}_n}, \quad H = \sum_{n \in \mathbb{Z}} \left(\frac{|u_{n+1} - u_n|^2}{h^2} - F(u_n, u_{n+1}) \right)$$

- These constraints *may provide* conservation of the power N

$$N = a \sum_{n \in \mathbb{Z}} |u_n|^2 + b \sum_{n \in \mathbb{Z}} \left(\bar{u}_{n+1} u_n + u_{n+1} \bar{u}_n \right)$$

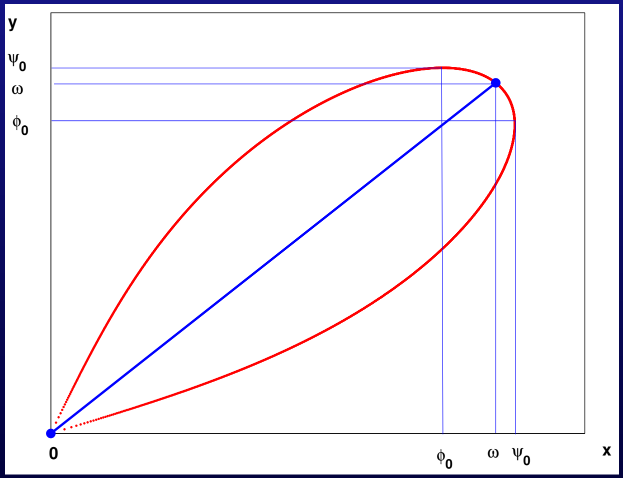
Continuous stationary solutions

Initial-value problem for real-valued solutions:

$$\begin{cases} (\phi_{n+1} - \phi_n)^2 = h^2 \omega \phi_n \phi_{n+1} - h^2 g(\phi_n, \phi_{n+1}), & n \in \mathbb{Z}, \\ \phi_0 = \varphi, \end{cases}$$

where

$$g(x, y) = \beta_1 x^2 y^2 + \beta_2 xy(x^2 + y^2) + \beta_3 (x^4 + y^4)$$



Solutions of the first-order map

- There exists a unique monotonically decreasing sequence $\{\phi_n\}_{n=0}^{\infty}$ for any $0 < \phi_0 < \sqrt{\omega}$.
- There exists a unique monotonically increasing sequence $\{\phi_n\}_{n=-\infty}^{0}$ for any $0 < \phi_0 < \sqrt{\omega}$.

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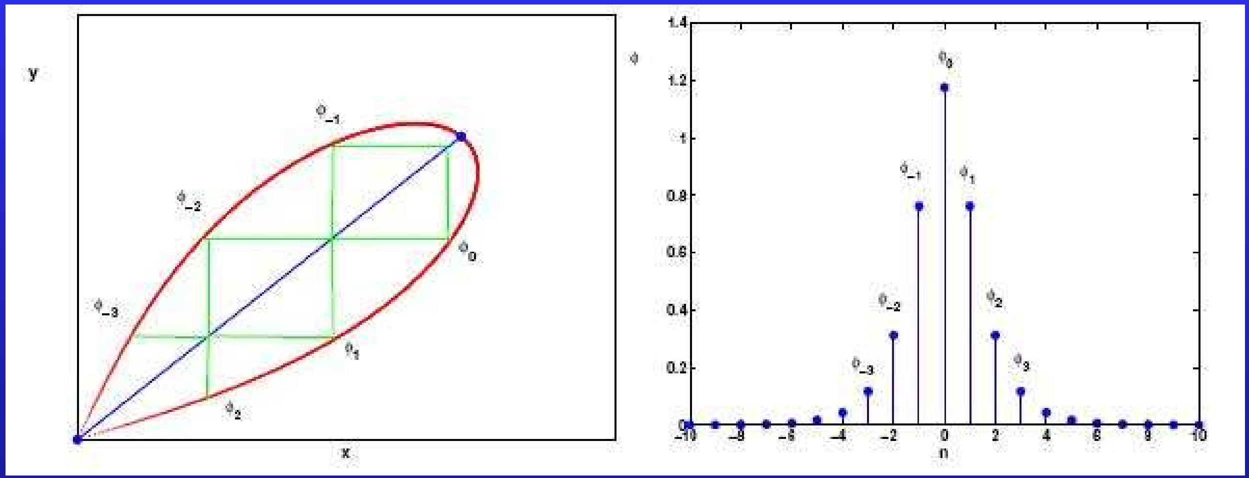
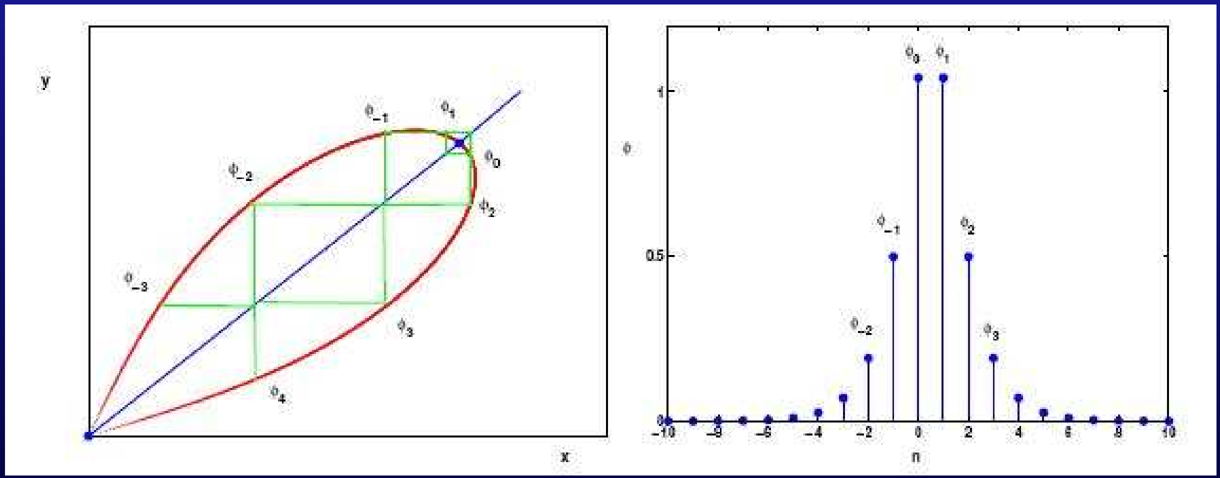
- There exists a unique single-humped sequence $S_{\text{on}} = \{\phi_n\}_{n=\mathbb{Z}}$ for $\phi_0 = \phi_{\max}$
- There exists a unique 2-site top single-humped sequence $S_{\text{off}} = \{\phi_n\}_{n=\mathbb{Z}}$ for $\phi_0 = \sqrt{\omega}$
- For any $\phi_0 \in (0, \phi_{\max}) \backslash \{S_{\text{on}}, S_{\text{off}}\}$, there exists a unique non-symmetric single-humped sequence $\{\phi_n\}_{n=\mathbb{Z}}$ with $\phi_k \neq \phi_m$ for all $k \neq m$.

Solutions of the first-order map

S_{on}:



S_{off}:



Traveling solutions

The reduction to the first-order map gives a *sufficient* condition for existence of the translationally invariant stationary solutions and a *necessary* condition for existence of traveling solutions near $c = 0$.

In other words, there exists $\phi(z) \in C^0(\mathbb{R})$ such that $\phi_n = \phi(hn - s)$ for $n \in \mathbb{Z}$ and $s \in \mathbb{R}$.

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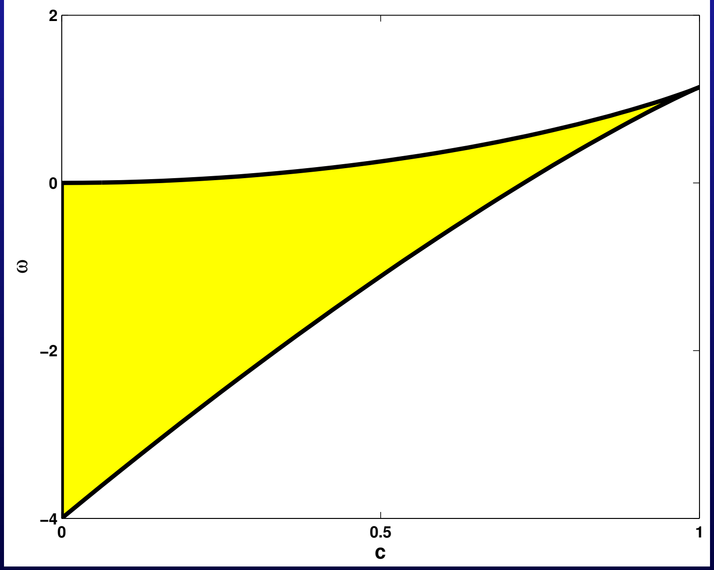
Another *necessary* condition for existence of traveling solutions is derived (P, Rothos, 2005) near the particular point:



$$\omega = \frac{\pi - 2}{h^2}, \; c = \frac{1}{h}$$

Reduction to the third-order ODE

Consider a transformation:

$$\phi(z) = \frac{\epsilon}{h}\Phi(\zeta)e^{\frac{i\pi z}{2h}},\ \zeta = \frac{\epsilon z}{h},\ c = \frac{1+\epsilon^2 V}{h},\ \omega = \frac{\pi - 2 + \epsilon^2\pi V + \epsilon^3\Omega}{h^2}$$

which results in the differential advance-delay equation:

$$i\left(\Phi(\zeta+\epsilon) - \Phi(\zeta-\epsilon) - 2\epsilon\Phi'(\zeta)\right) = \epsilon^3\left(2iV\Phi'(\zeta) + \Omega\Phi(\zeta)\right) - \epsilon^2 f(...)$$

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Apply Taylor series expansions:

$$\Phi(\zeta+\epsilon) - \Phi(\zeta-\epsilon) - 2\epsilon\Phi'(\zeta) = \frac{\epsilon^3}{3}\Phi'''(\zeta) + \mathrm{O}(\epsilon^5),$$

$$f(\ldots) = (\alpha_1 + 2\alpha_4 - 2\alpha_5 - 2\alpha_6 + \alpha_7)\,|\Phi|^2\Phi + \mathrm{O}(\epsilon).$$

Reduction to the third-order ODE

Since no single-humped localized solutions exist in

$$\frac{i}{3}\Phi''' - 2iV\Phi' - \Omega\Phi = |\Phi|^2\Phi,$$

the necessary condition for existence of traveling solutions is

$$\alpha_1 + 2\alpha_4 - 2\alpha_5 - 2\alpha_6 + \alpha_7 = 0.$$

The truncated third-order ODE is

$$\frac{i}{3}\Phi''' - 2iV\Phi' - \Omega\Phi + 2i|\Phi|^2\Phi' + i\gamma\Phi(|\Phi|^2)' = 0,$$

where γ is parameter.

Translationally invariant dNLS models

Parametrization of the dNLS model which gives translationally invariant solutions at $c = 0$ and $c = 1/h$:

$$\alpha_1 = 2\alpha_6, \ \alpha_4 = \alpha_5 = \alpha_6, \ \alpha_7 = 0, \ \alpha_{10} = \alpha_8 - \alpha_9,$$

subject to the normalization constraint:

$$\alpha_2 + \alpha_3 + 4\alpha_6 + 2\alpha_8 = 1.$$

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Additional conserved quantities:

- Conservation of power N gives four *one-parameter* models
- Conservation of density flux gives a *two-parameter* model

Open questions

Traveling solutions of the third-order ODE:

- $\gamma = 0$ - Hirota equation with 2-parameter solutions
- $\gamma = 1$ - Sasa-Satsuma equation with 2-parameter solutions
- $\gamma > -1$ - exact 1-parameter solutions (embedded solitons)

Can we prove *persistence* of any of these solutions in the full differential advance-delay equation?

Numerical approximation of traveling solutions is a work in progress.