Traveling waves in nonlinear lattices

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Fields Institute, April 28-29, 2006

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Continuous ϕ^4 model

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where $|c| < 1$ and $s \in \mathbb{R}$.

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"Standard" (on-site) discretisation:

$$\ddot{u}_n = \frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} + u_n(1-u_n^2), \quad n \in \mathbb{Z}$$

Does the discrete model have traveling kink solutions?

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Continuous NLS model

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where $\omega \in \mathbb{R}_+$ and $(c, s, \theta) \in \mathbb{R}^3$.

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Does the discrete model have stationary solitary wave solutions?

Reductions for traveling waves

Traveling waves

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\begin{aligned}
u_1(t) &= u_0(t-\tau), \\
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The differential advanced-delay equation

$$c^2\phi'' = \frac{\phi(z+h) - 2\phi(z) + \phi(z-h)}{h^2} + \phi(1-\phi^2)$$

Obstacles on existence

Classical solutions $\phi(z)$ on $z \in \mathbb{R}$

- $\phi(z)$ is $C^0(\mathbb{R})$ if $c = 0$
- $\phi(z)$ is $C^2(\mathbb{R})$ if $c \neq 0$
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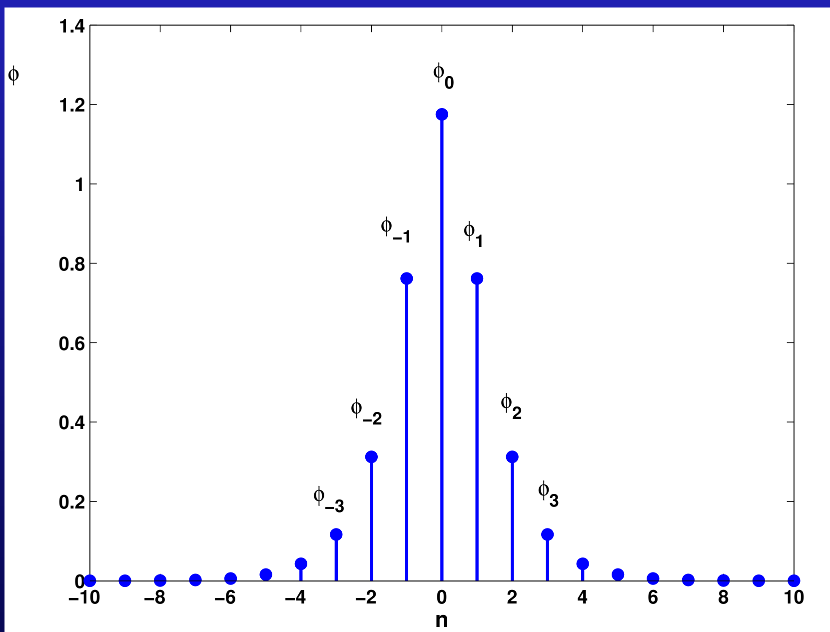
Stationary solutions ($c = 0$) in standard discretizations:

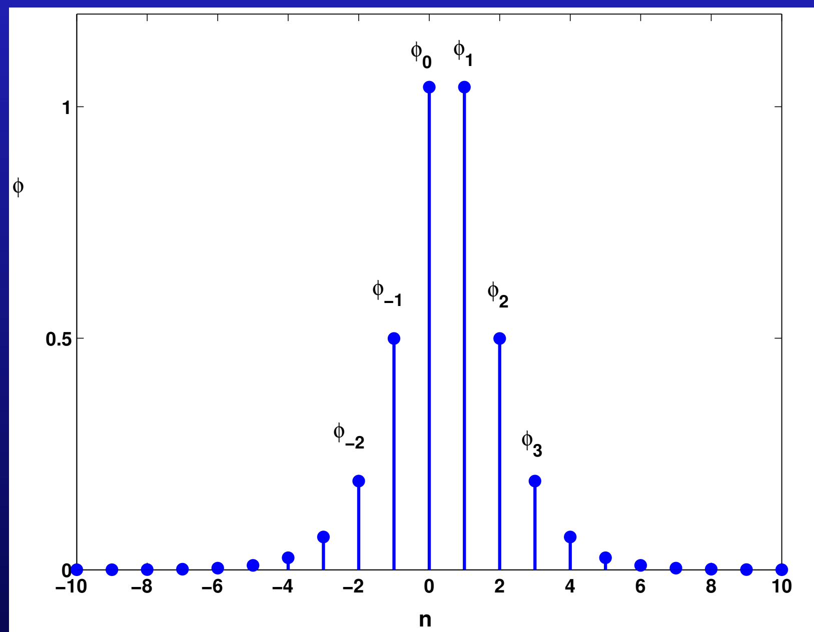
- $\phi(z)$ is piecewise constant on $z \in \mathbb{R}$
- $\phi_n = \phi(nh)$ is odd either about $n = 0$ or about the midpoint between $n = 0$ and $n = 1$
- No continuous deformation exists between two particular solutions (Peierls–Nabarro potential)

Example of stationary solutions

Stationary solutions in the standard discrete NLS model

$$\frac{\phi_{n+1} - 2\phi_n + \phi_{n-1}}{h^2} + \phi_n^3 - \phi_n = 0, \quad n \in \mathbb{Z}$$





General and exceptional discretizations

General discrete ϕ^4 model:

$$\ddot{u}_n = \frac{u_{n+1} - 2u_n + u_{n-1}}{h^2} + f(u_{n-1}, u_n, u_{n+1})$$

where

P1 (continuity) $f(u,u,u) = u(1-u^2)$

P2 (symmetry) $f(v,u,w) = f(w,u,v)$

P3 $f(v,u,w)$ is independent on h

P4 $f(v,u,w) = u - Q(v,u,w)$, where $Q = O(3)$

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Exceptional nonlinearities are those that support continuous stationary solutions with $c = 0$ and $\phi \in C^0(\mathbb{R})$

Examples of exceptional discretizations

Tovbis (1997) $f = (u_{n+1} + u_{n-1}) \left(1 - u_n^2\right),$

Speight (1997) $f = (2u_n + u_{n+1}) \left(3 - u_n^2 - u_n u_{n+1} - u_{n+1}^2\right) + \{n+1 \to n\},$

Kevrekidis (2003) $f = (u_{n+1} + u_{n-1}) \left(2 - u_{n+1}^2 - u_{n-1}^2\right).$

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Exceptional stationary solutions:

- The stationary solution has a translation parameter, e.g.

$$u_n = \tanh\left(a(hn - s)\right), \quad a = \frac{1}{h}\arcsin(h/2)$$

- Radiation from moving kinks is reduced

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- Q: Can we characterize all possible bifurcations of steadily moving kinks?
- A: YES, when the center manifold is finite-dimensional (when $c \neq 0$).

Families of exceptional discretizations

Consider the second-order difference map for stationary solutions

$$\frac{\phi_{n+1} - 2\phi_n + \phi_{n-1}}{h^2} + f(\phi_{n-1}, \phi_n, \phi_{n+1}) = 0$$

and reduce the problem to the first-order difference map

$$E = \frac{\phi_{n+1} - \phi_n}{h} - g(\phi_n, \phi_{n+1}) = \mathrm{const}$$

Such discretizations with *polynomial* functions $g(\phi_n, \phi_{n+1})$ exist for *exceptional* polynomial functions $f(\phi_{n-1}, \phi_n, \phi_{n+1})$.

Continuous stationary solutions

Theorem[Speight, 1999]: Let $g(\phi_n, \phi_{n+1})$ be a polynomial such that $g(\phi, \phi) = \frac{1}{2}(1 - \phi^2)$. Then, for any $-1 < \phi_0 < 1$, there exists a unique monotonic sequence $\{\phi_n\}_{n\in\mathbb{Z}}$ such that

$$\phi_n < \phi_{n+1}, \qquad \lim_{n\to\pm\infty} \phi_n = \pm 1$$

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Corollary: There exists a $C^0(\mathbb{R})$ monotonic kink solution $\phi(z - s)$ of the advance-delay equation

$$\frac{\phi(z-h) - 2\phi(z) + \phi(z+h)}{h^2} + f(\phi(z-h), \phi(z), \phi(z+h)) = 0,$$

such that $\lim_{z \to \pm\infty} \phi(z) = \pm 1$ and $\phi_0 = \phi(-s)$.

Traveling solutions with $c \neq 0$?

Solutions $\phi(z) = e^{\lambda z}$ of the linearized equation at $\phi = 0$:

$$c^2 \phi''(z) = \frac{\phi(z+h) - 2\phi(z) + \phi(z-h)}{h^2} + \phi(z)$$

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Dispersion relation for $\Lambda = \lambda h$:

$$D(\Lambda; c, h) = 2\left(\cosh \Lambda - 1\right) + h^2 - c^2 \Lambda^2 = 0$$

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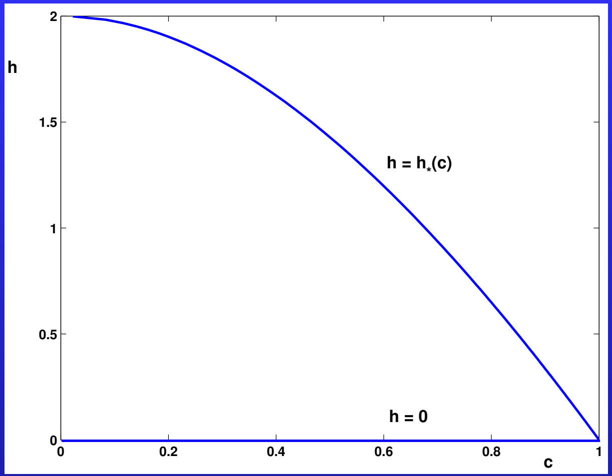
Roots on the imaginary axis $\Lambda = 2iK$, $K \in \mathbb{R}$

$$\sin^2 K = \frac{h^2}{4} + c^2 K^2$$

Parameter plane (c, h) and bifurcations



Three bifurcations:
$c = 0,\ 0 < h < 2$
$h = 0,\ 0 < c < 1$
$h = h_*(c),\ 0 < c < 1$

- $c = 0,\ 0 < h < 2$: All roots of K are real and simple.
- $h = 0,\ 0 < c < 1$: Double zero coexists with finitely many pairs of real roots K.
- $h = h_*(c),\ 0 < c < 1$: One pair of double real (non-zero) roots K exist (1:1 resonant Hopf bifurcation).

Bifurcation at $h = 0$

Differential advance-delay equation:

$$c^2\phi''(z) = \frac{\phi(z+h) - 2\phi(z) + \phi(z-h)}{h^2} + \phi(z) \\ - Q\left(\phi(z-h), \phi(z), \phi(z+h)\right)$$

Formal perturbation expansion:

$$\phi(z+h) - 2\phi(z) + \phi(z+h) = h^2\phi''(z) + \sum_{n=2}^{\infty} \frac{2}{(2n)!}\phi^{(2n)}(z)h^{2n}$$

and

$$\phi(z) = \sum_{n=0}^{\infty} h^{2n}\phi_{2n}(z)$$

Beyond all order asymptotics

At the leading order $\mathcal{O}(h^0)$:

$$(1-c^2)\phi_0''+\phi_0(1-\phi_0^2) = 0; \quad \Rightarrow \quad \phi_0(z) = \tanh\theta, \quad \theta = \frac{z}{2\sqrt{1-c^2}}.$$

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At the higher orders $\mathcal{O}(h^{2n})$:

$$\mathcal{L}\phi_{2n} = (\text{odd inhomogeneous terms})$$

where $\mathcal{L} = -\frac{d^2}{d\theta^2} + 4 - 6\,\mathrm{sech}^2\,\theta$ with $\mathcal{L}\mathrm{sech}^2\,\theta = 0$.

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To all orders, the perturbation expansion exists and

$$\phi(z) \to \pm 1 \quad \text{as} \quad |z| \to \infty.$$

Does it exist beyond all orders of the perturbation expansion?

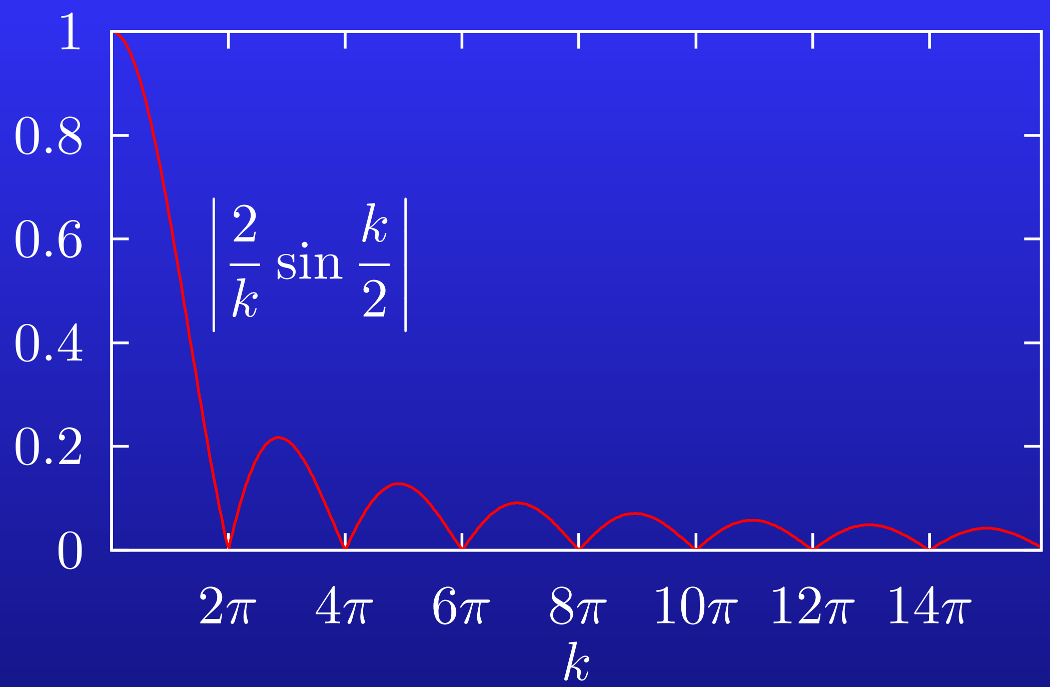
Beyond all order asymptotics

Fourier oscillatory modes:

$$\phi(z) = \pm 1 + \epsilon e^{ikz/h}$$

has wavenumber k where

$$\sin^2 k = c^2 k^2 \text{ as } h \to 0.$$



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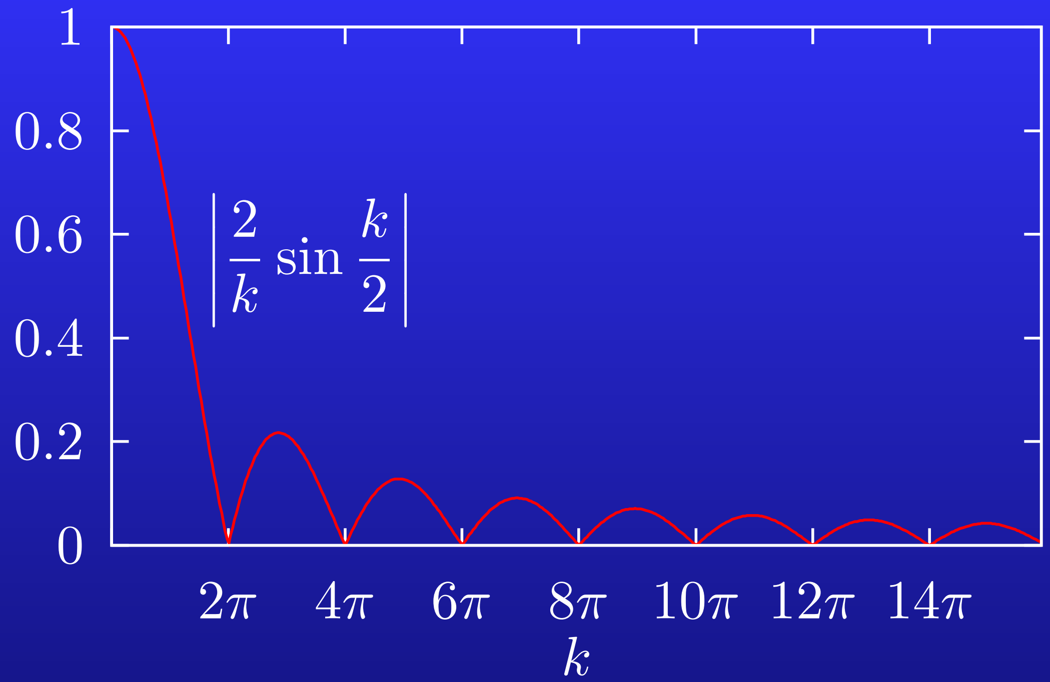
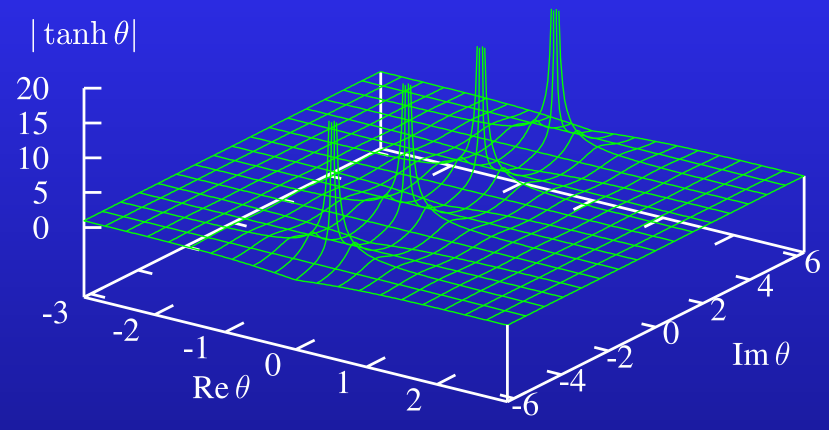


Fourier modes do not occur in power series of h.

Inner equation

Kruskal–Segur (1991):
Continue the solution into the complex plane and study Fourier modes near the poles of the regular perturbation expansion



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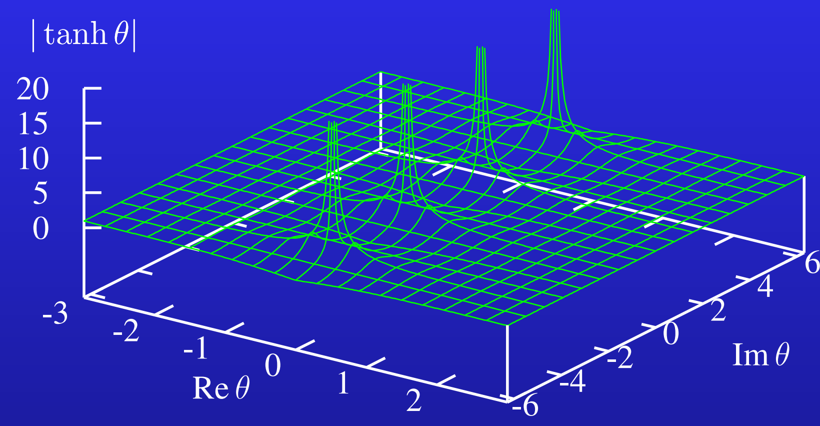


Scaling transformation $z = h\zeta + i\pi\sqrt{1-c^2}$ and $\phi(z) = \frac{1}{h}\psi(\zeta)$ leads to the inner equation:

$$c^2\psi''(\zeta) = \psi(\zeta+1) - 2\psi(\zeta) + \psi(\zeta-1)$$

$$- Q\left(\psi(\zeta-1), \psi(\zeta), \psi(\zeta+1)\right) + \frac{h^2}{2}\psi(\zeta).$$

Inner asymptotic series

Let the solution $\psi(\zeta)$ be expanded in powers of h^2:

$$\psi(\zeta) = \psi_0(\zeta) + \sum_{n=1}^{\infty} h^{2n} \psi_{2n}(\zeta).$$

Let the leading-order solution $\psi_0(\zeta)$ be expanded in inverse power series of ζ:

$$\psi_0(\zeta) = \sum_{n=0}^{\infty} \frac{a_{2n}}{\zeta^{2n+1}}, \qquad a_0 = 2\sqrt{1-c^2}.$$

Theorem (Tovbis, 2000): If the inverse power series for $\psi_0(\zeta)$ diverges, the formal perturbation expansion for $\phi(z)$ diverges and some Fourier modes are non-zero beyond the formal expansion.

Borel-Laplace transform

Let $\psi_0(\zeta)$ be the Laplace transform of $V_0(p)$:

$$\psi_0(\zeta) = \int_\gamma V_0(p) e^{-p\zeta} dp$$

The resulting integral equation,

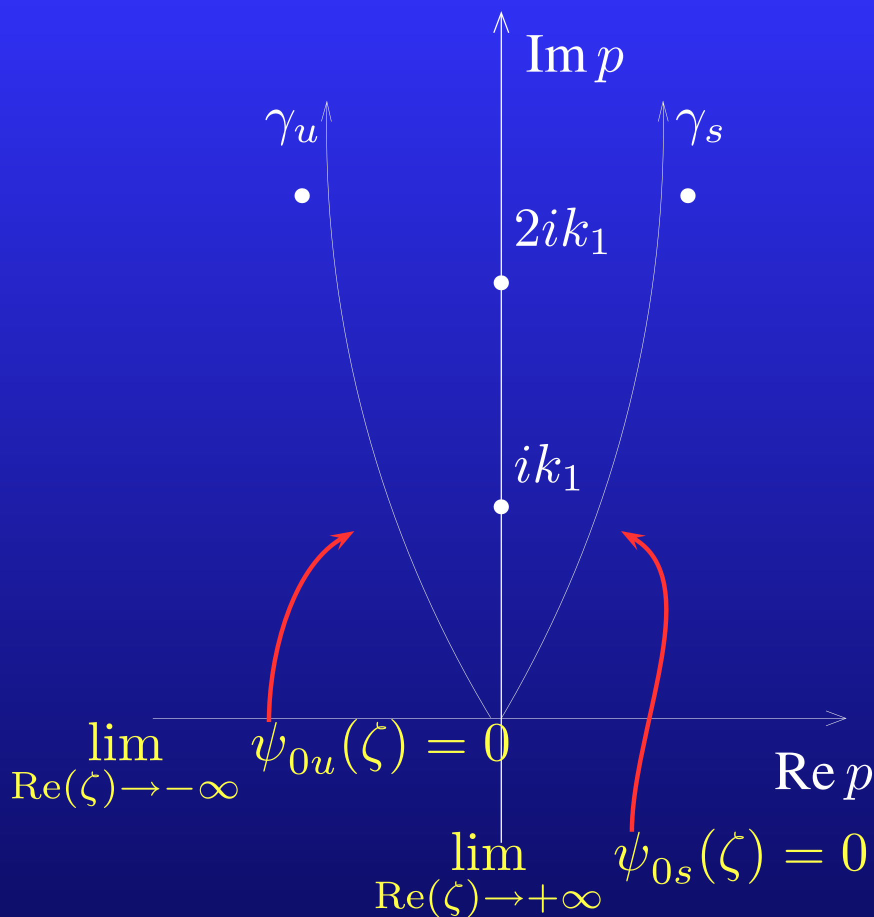
numerical coefficients

$$\left(4\sinh^2\frac{p}{2} - c^2p^2\right) V_0(p) = \sum_{\alpha_1,\alpha_2,\alpha_3} a_{\alpha_1,\alpha_2,\alpha_3}\ e^{\alpha_1 p} V_0(p) * e^{\alpha_2 p} V_0(p) * e^{\alpha_3 p}$$

Power series solution:

$$V_0(p) = \sum_{n=0}^{\infty} v_{2n} p^{2n}, \qquad v_0 = 2\sqrt{1-c^2}.$$

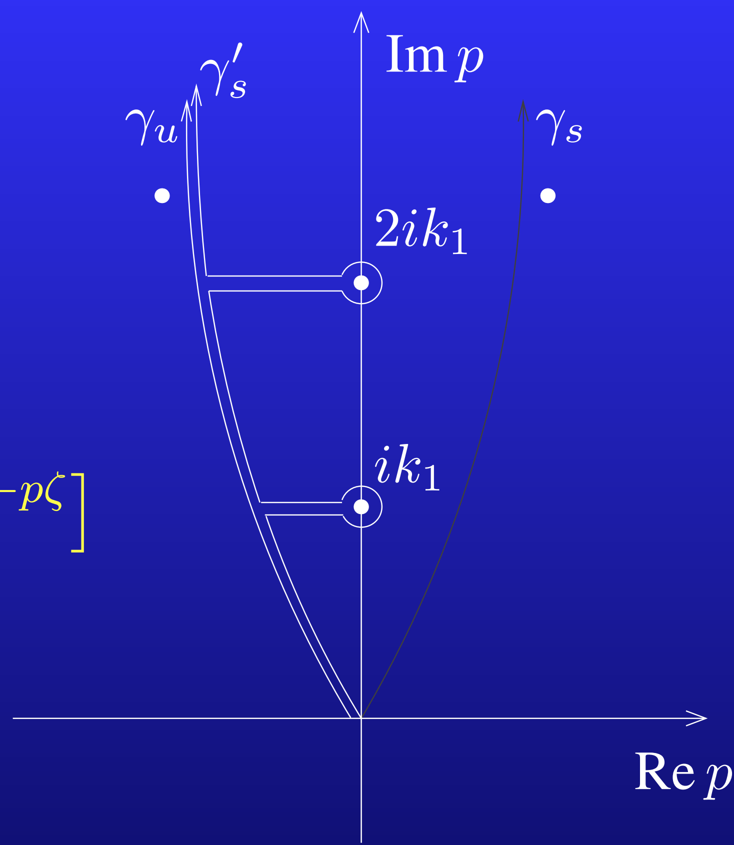
Singularities of $V_0(p)$



Singularities of $V_0(p)$

The distance between the stable and unstable manifold:

$$\psi_{0s}(\zeta)-\psi_{0u}(\zeta) = 2\pi i \sum \mathrm{Res}\ \left[V_0(p)e^{-p\zeta}\right]$$



The first pole of $V_0(p)$

Let $p = ik_1$ be the nearest singularity to $p = 0$. Then, $V_0(p)$ has the leading order behavior near $p = ik_1$:

$$V_0(p) \to \frac{k_1^2 K_1(c)}{(p^2 + k_1^2)},$$

where $K_1(c)$ is referred to as the *Stokes constant*.

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Then

$$\psi_{0s}(\zeta) - \psi_{0u}(\zeta) = [\pi k_1 K_1(c) + \mathcal{O}(1/\zeta)]e^{-ik_1\zeta}$$

Residue of the first pole $p = ik_1$ can be deduced from a power-series solution for $V_0(p)$ at $p = 0$, which converges for $|p| < k_1$.

Numerical computations of $K_1(c)$

- Expand $V_0(p)$ near the pole $p \to \pm i k_1$:

$$V_0(p) \to K_1(c) \sum_{n=0}^{\infty} (-1)^n k_1^{-2n} p^{2n}$$

- Expand $V_0(p)$ in the power series at $p = 0$:

$$V_0(p) = \sum_{n=0}^{\infty} v_{2n} p^{2n}, \qquad v_0 = 2\sqrt{1 - c^2},$$

where the coefficients are obtained from a recurrence relation.

- Then $K_1(c)$ can be obtained as a numerical limit:

$$K_1(c) = \lim_{n \to \infty} w_n, \qquad w_n = (-1)^n k_1^{2n} v_{2n}$$

Convergence of the algorithm

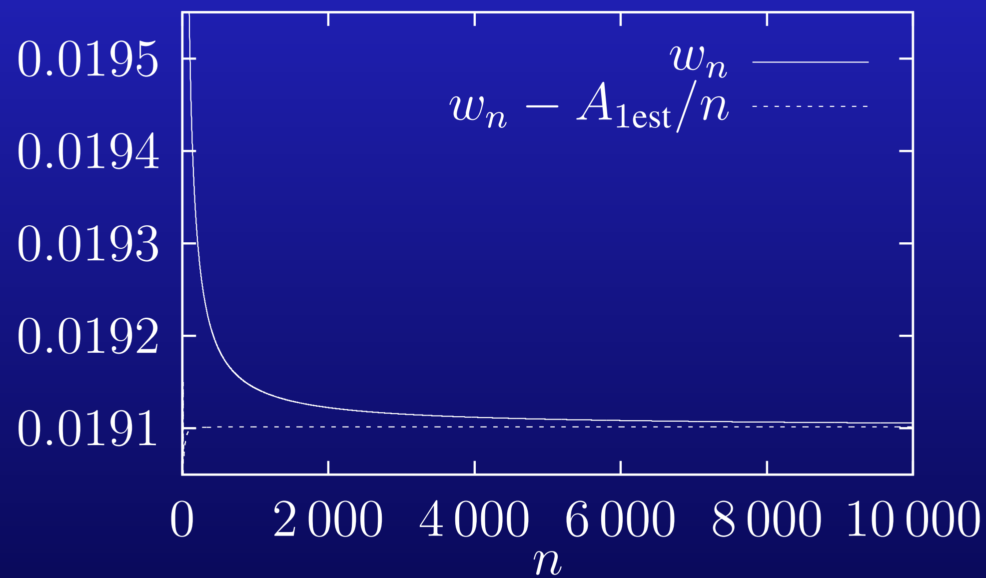
Convergence of w_n is slow:

$$w_n = K_1(c) + \frac{A(c)}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)$$

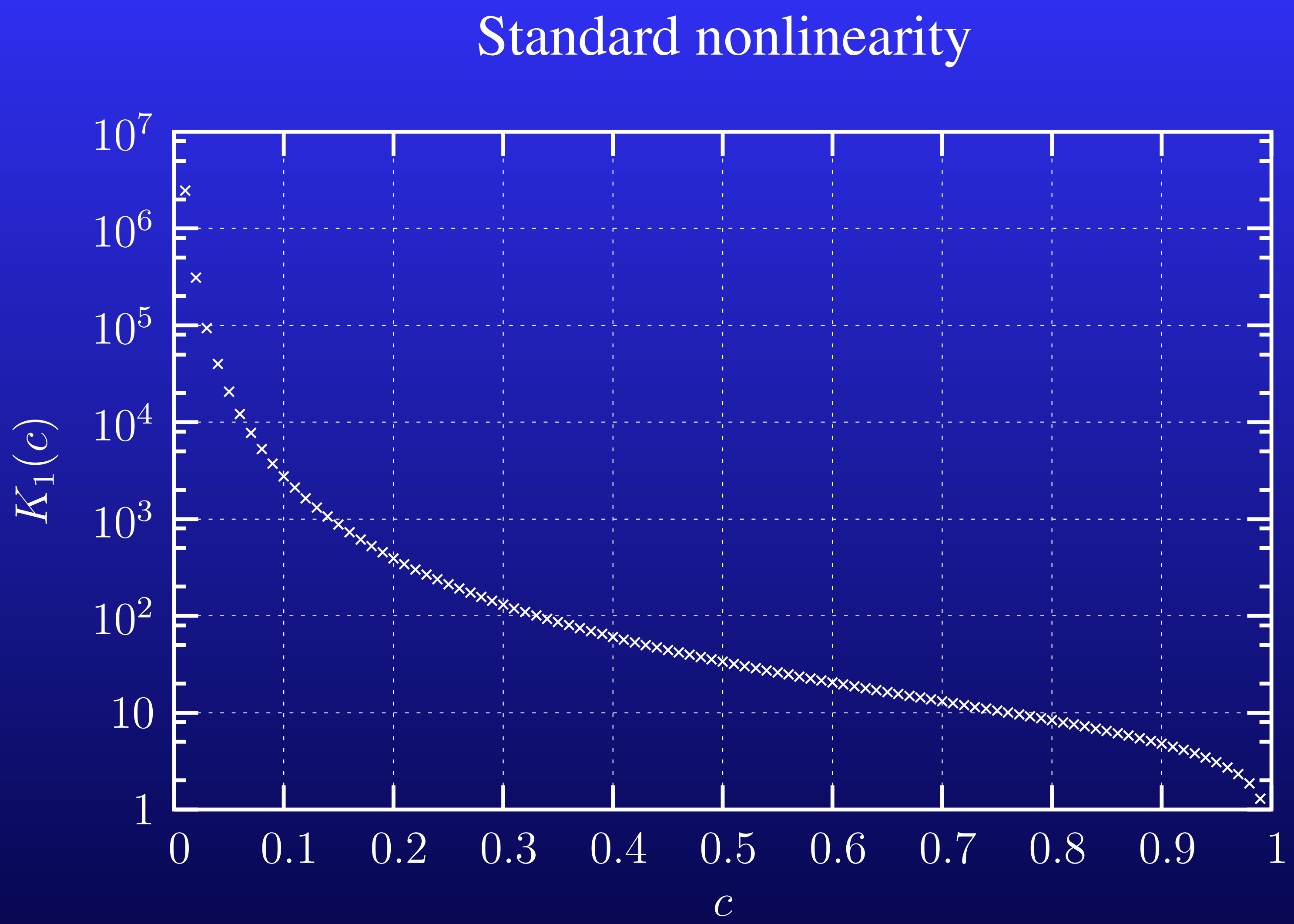
The error estimate of the numerical limit:

$$A_{\text{est}} = -n^2(w_n - w_{n-1})$$

Kevrekidis' model with $c = 0.5$:

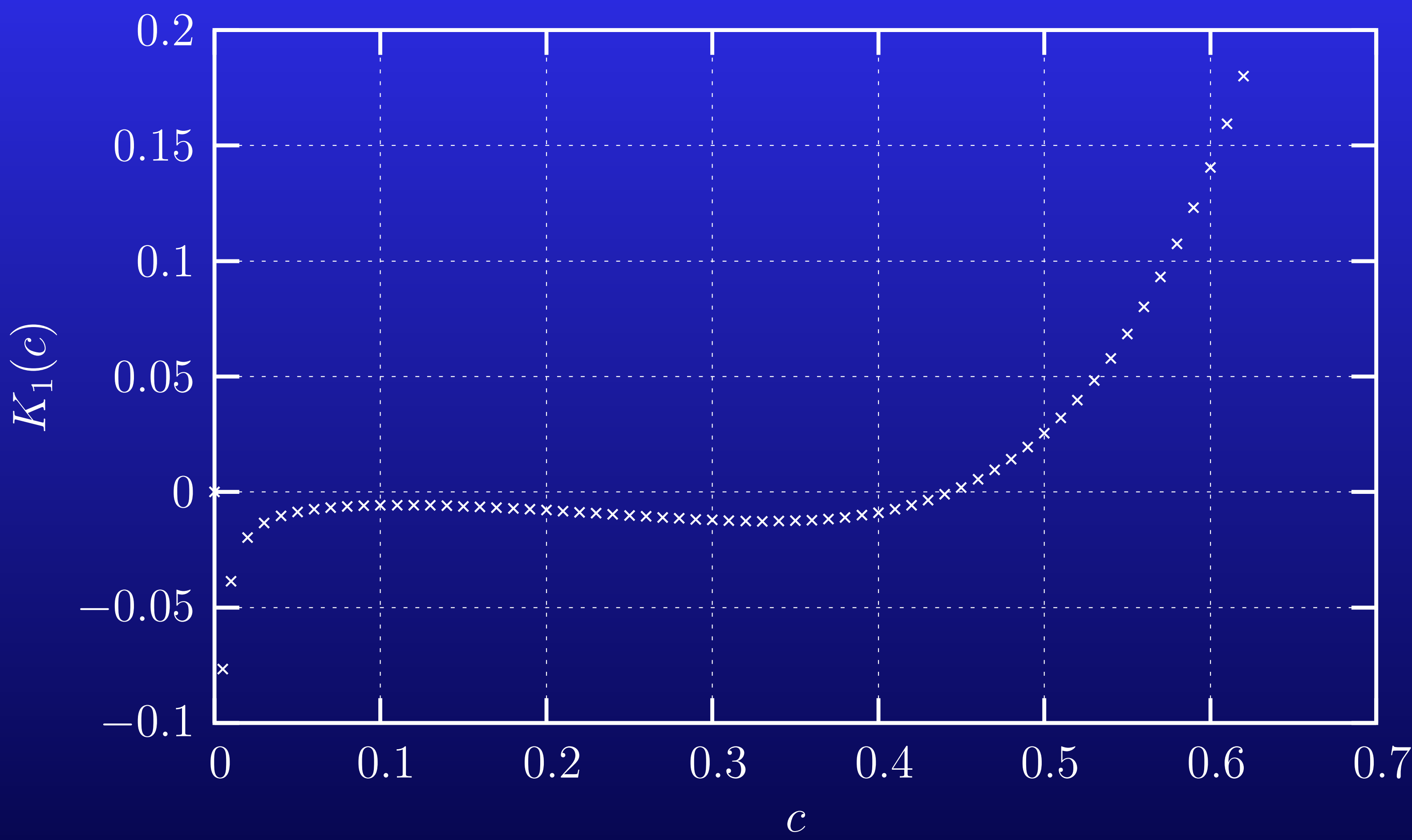


Stokes Constant



Stokes Constant

Speight's discretization



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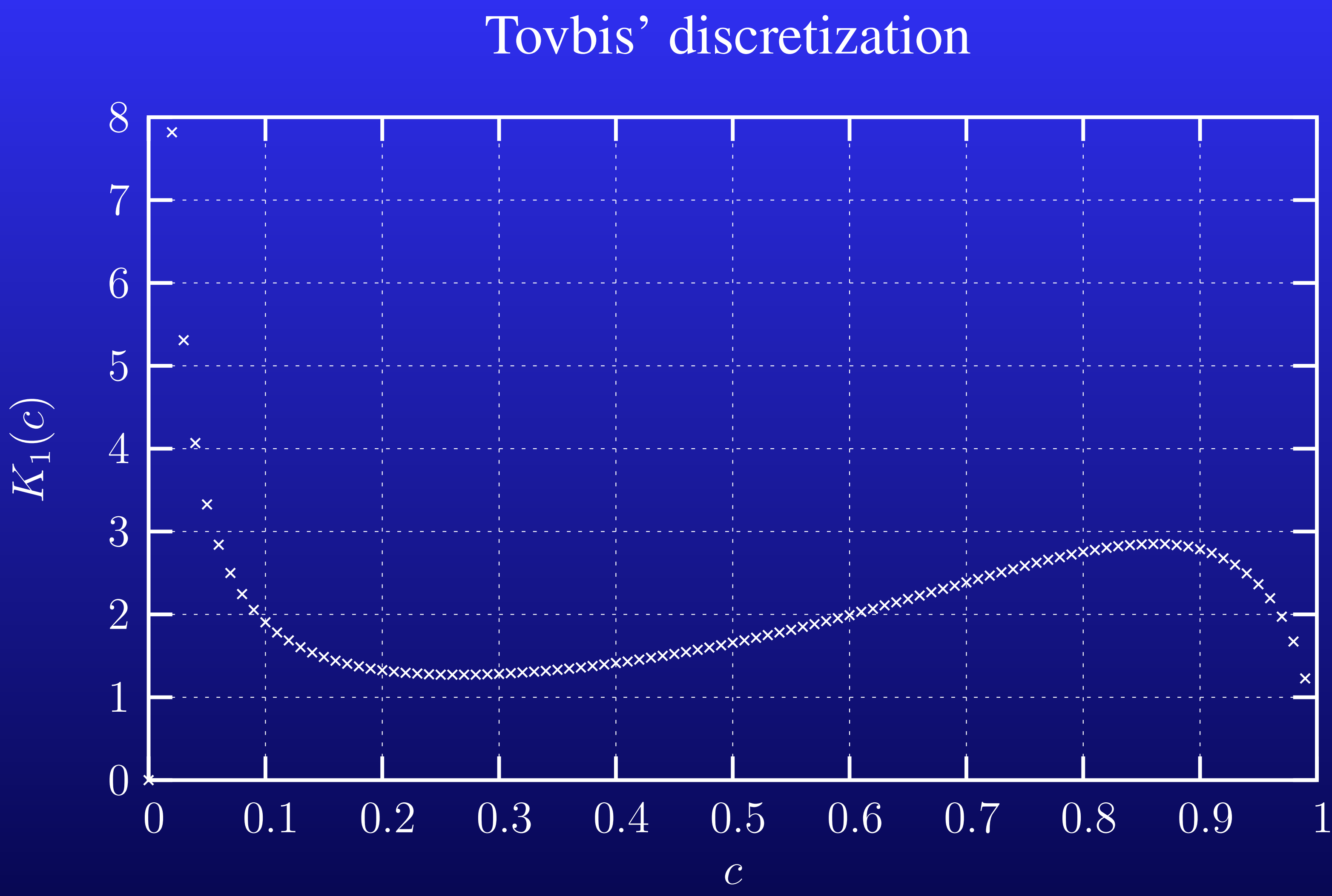

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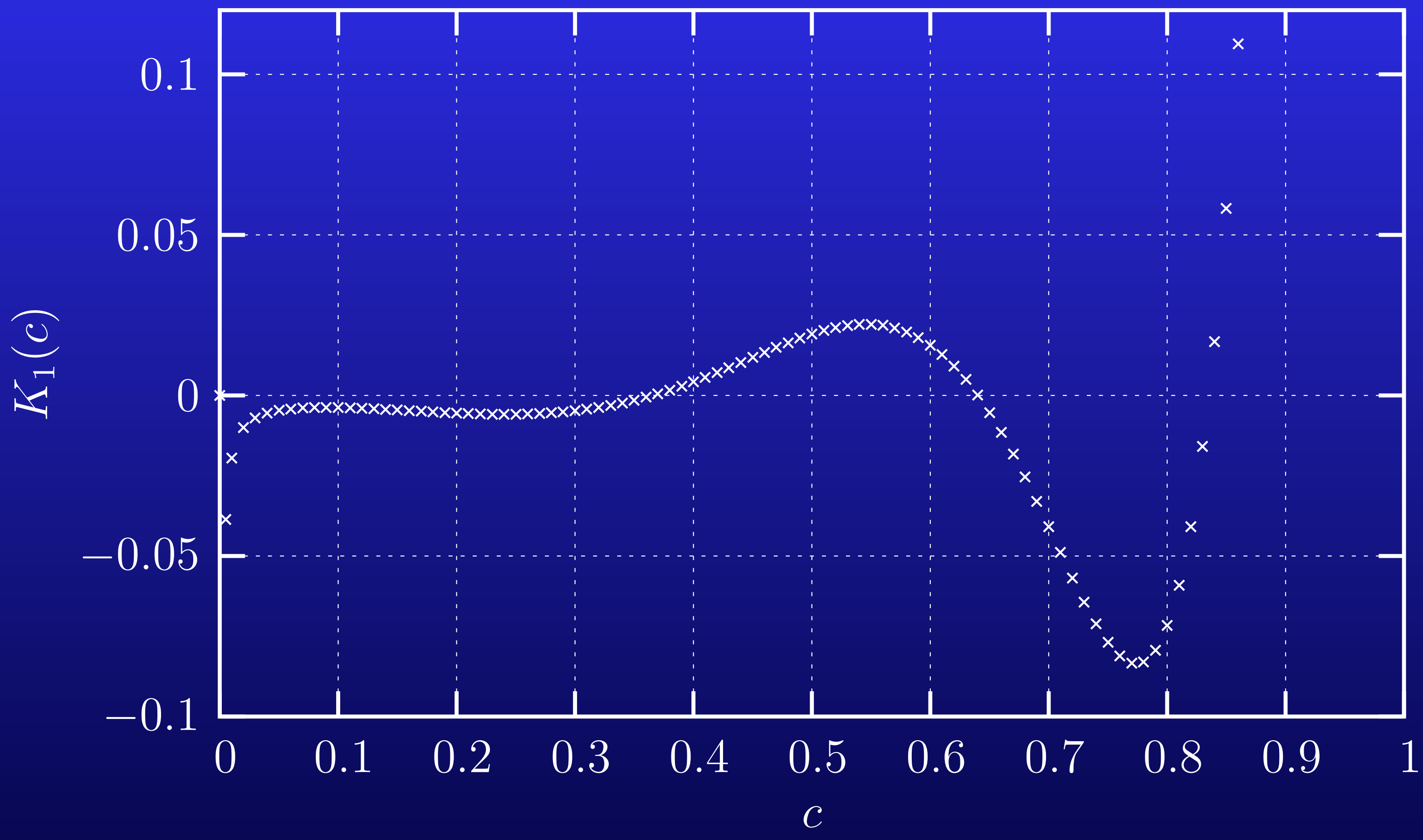
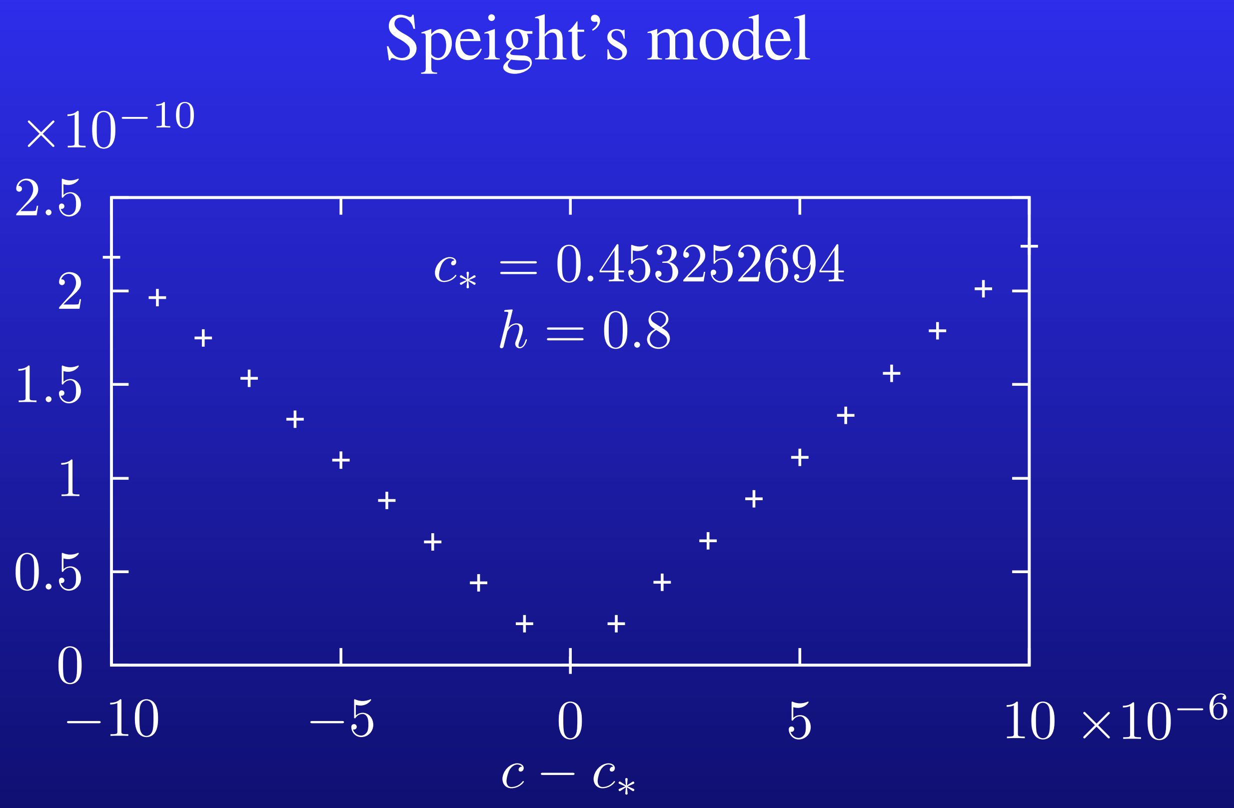


Bifurcations of traveling kinks on (c, h)

- Zeros $c = c_*$ of $K_1(c)$ lie in the region $c > 0.22$, where only one pair $p = \pm i k_1$ is purely imaginary.
- One can expect a bifurcation of the one-parameter curve on the (h, c) plane that passes through the point $(0, c_*)$.
- Numerical analysis of the bifurcation:
  - solve the differential advance-delay equation on $z \in [-L, L]$ where $L = 100$
  - subject to the anti-periodic boundary conditions $\phi(L) = -\phi(-L)$
  - by using the iterative Newton's method with the continuum kink as starting guess,
  - the eight-order finite-difference approximation to the second derivative with the step size $\Delta z = h/10$.

Numerical analysis of the bifurcation



The average of $[\phi(z) - \phi_{\text{ave}}]^2$ is computed over the the interval $z \in [L-20, L]$ for fixed values of parameter c

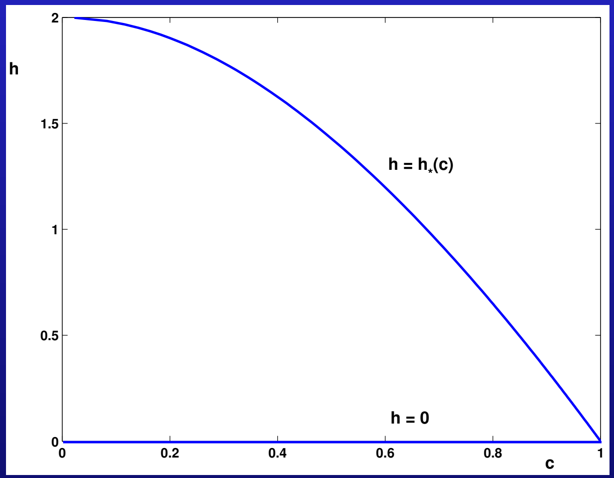
Bifurcation at $h = 0, c = 1$

Differential advance-delay equation (inner form):

$$c^2\phi'' = \phi(\zeta+1)-2\phi(\zeta)+\phi(\zeta-1)+h^2\phi-h^2Q(\phi(\zeta-1),\phi(\zeta),\phi(\zeta+1)),$$

where $\zeta = z/h$.

Near the point
$h = 0, c = 1$:
$c^2 = 1 + \epsilon\gamma$
$h^2 = \epsilon^2\tau$
$\zeta_1 = \sqrt{\epsilon}\zeta$



Truncated normal form for bifurcation:

$$\frac{1}{12}\phi^{(\mathrm{iv})}(\zeta_1) - \gamma\phi''(\zeta_1) + \tau\phi(\zeta_1)(1-\phi^2(\zeta_1)) = 0.$$

Numerical analysis of heteroclinic orbits

Truncated normal form:

$$\phi^{(\mathrm{iv})} + \sigma\phi'' + \phi - \phi^3 = 0,$$

where $\sigma = -\sqrt{12}\gamma/\sqrt{\tau}$ and $\phi = \phi(t)$.

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Linearization at $\phi = \pm 1$ gives pairs of eigenvalues $(\lambda_0, -\lambda_0)$ and $(i\omega_0, -i\omega_0)$ with the one-dimensional unstable manifold:

$$\lim_{t \to -\infty} \phi_u(t) = -1, \qquad \lim_{t \to -\infty} (\phi_u(t) + 1)\, e^{-\lambda_0 t} = C_0$$

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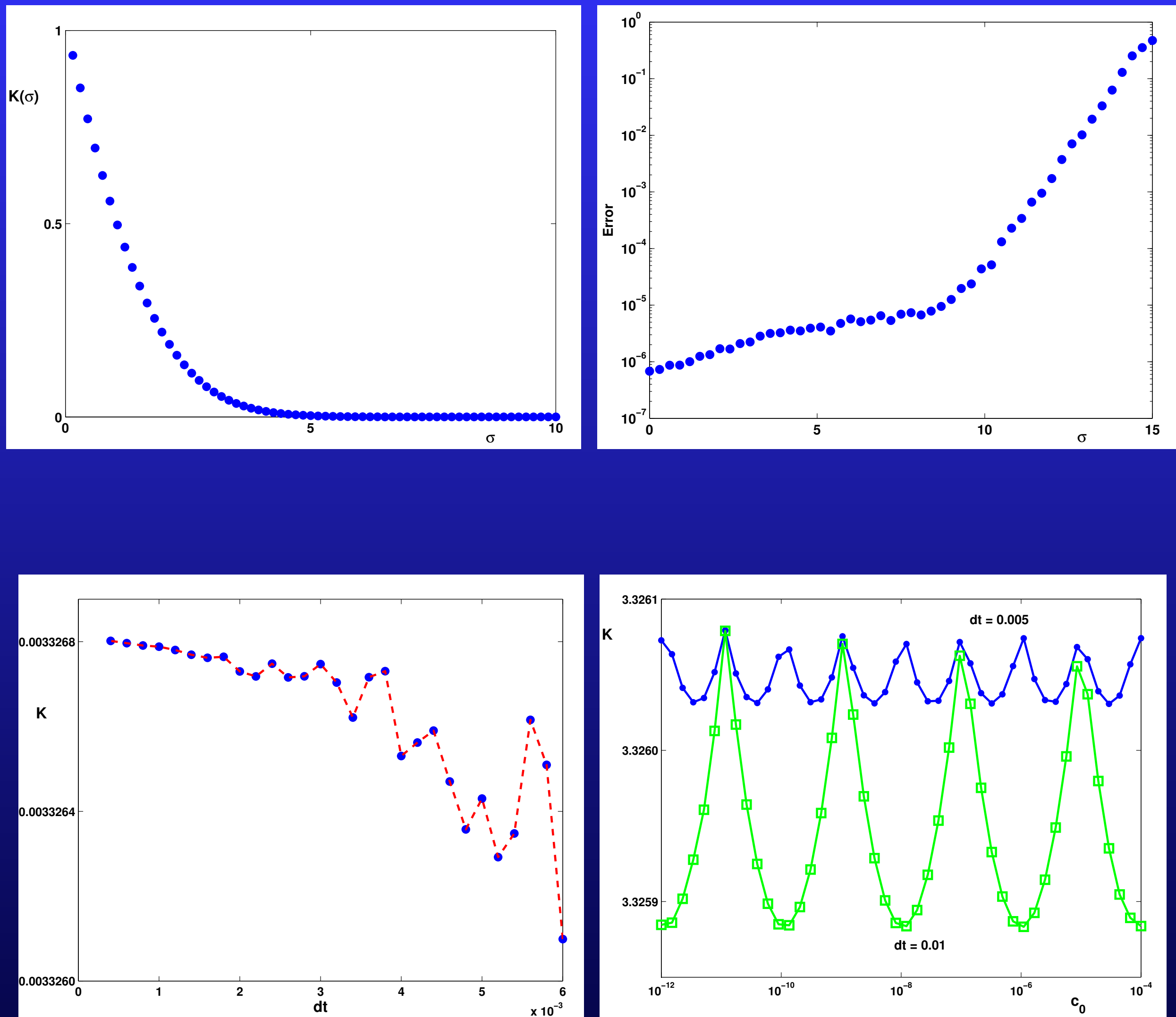
The kink solution is odd in $t \in \mathbb{R}$ (up to translational invariance). Iterating the initial-value problem along the unstable manifold from $t = 0$ to $t = t_0$ where $\phi(t_0) = 0$, one can compute the split function $K(\sigma) = \phi''(t_0)$, which may depend on numerical factors C_0 and Δt.

Numerical results on $K(\sigma)$



No bifurcations of kinks occur from the point $h = 0, c = 1$.

Conclusions

- Existence of continuous stationary kinks at $c = 0$ is not sufficient for existence of traveling kinks at $c \neq 0$
- Bifurcations of traveling kinks may occur at isolated velocities with $0 < c < 1$ (e.g. in the numerical analysis of the Speight's and Kevrekidis' exceptional nonlinearities)
- No bifurcations of traveling kinks occur from the point $c = 1$
- It is problematic to consider asymptotic expansions in powers of c^2 and the bifurcation of traveling kinks from the point $c = 0$