Validity of the weakly-nonlinear solution of the Boussinesq-Ostrovsky equation

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References:

K.R. Khusnutdinova, K. R. Moore, and D.P., arXiv: 1308.2611 (2013)
D.P. and D.V. Ponomarev, Z. angew. Math. Phys., in press (2013)
D.P. and G. Schneider, Non. Diff. Eqs. Appl. **20**, 1277 (2013)

Boussinesq-Ostrovsky equation

We are concerned here in the regularized Boussinesq-Ostrovsky equation

$$U_{tt} - U_{xx} = \epsilon \left(\frac{1}{2}(U^2)_{xx} + U_{ttxx} - \gamma U \right),$$

where ϵ is a small parameter and γ is the rotation-modification parameter. When $\gamma = 0$, this is just the regularlized Boussinesq equation.

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The weakly-nonlinear solution is given by the two counter-propagating waves satisfying the Korteweg–de Vries (Ostrovsky) equations.

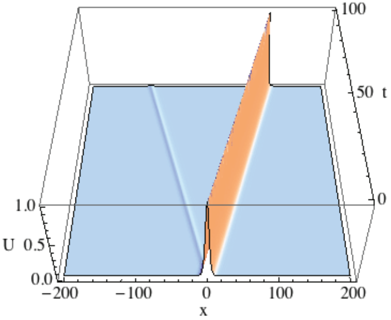

Figure: Evolution of the weakly nonlinear solution U for $\epsilon = 0.1$.

Justification of the weakly-nonlinear solution for $\gamma = 0$

- First results are due to Craig (1985), Schneider (1998), Schneider & Wayne (2000,2002), Ben Youssef & Colin (2000), and Lannes (2003)
- On the infinite line, the error was controlled in Sobolev space by Wayne & Wright (2002). The first-order correction satisfying the linearized KdV equation was incorporated in the weakly–nonlinear solution.
- On the infinite line and in the periodic domain, the error was controlled with energy estimates by Bona, Colin, & Lannes (2005).
- Closed-form expressions for the first-order corrections were found by Khusnutdinova & Moore (2011,2012), without the use of the linearized KdV equation. Numerical results were provided without justification or convergence analysis.

Goals and novelty of our work

- We deal with the periodic solutions both for $\gamma = 0$ and $\gamma \neq 0$. We incorporate the closed-form solutions into convergence analysis and illustrate the rates of convergence with numerical approximations.
- Compared to Wayne & Wright (2002), we work in the natural energy space, instead of Sobolev spaces of higher regularity.
- Compared to Bona, Colin, & Lannes (2005), we incorporate the nonzero mean term of the periodic solutions and show that this term does not reduce the validity of the weakly-nonlinear approximation.
- Compared to Khusnutdinova & Moore (2012), we show that the first-order corrections must satisfy the linearized Ostrovsky equations to preserve the superior accuracy of the weakly-nonlinear approximation.
- We do not use the concepts of asymptotic integrability or commuting flows of the KdV hierarchy to analyze validity of the weakly-nonlinear solution, compared to the earlier works of Kraenkel, Manna, and Pereira (1995, 1997) and Kodama & Mikhailov (1997).

We consider the Cauchy problem for the regularized Boussinesq-Ostrovsky equation

$$U_{tt} - U_{xx} = \epsilon \left(\frac{1}{2}(U^2)_{xx} + U_{ttxx} - \gamma U \right),$$

subject to the initial values

$$U|_{t=0} = F(x), \quad U_t|_{t=0} = V(x),$$

in the period domain on $[-L, L]$.

Lemma

Fix $s > \frac{1}{2}$. For any $(F, V) \in H^s_{\mathrm{per}}(-L, L) \times H^s_{\mathrm{per}}(-L, L)$, there exists an ϵ-independent $t_0 > 0$ and a unique solution $U(t) \in C^1([0, t_0], H^s_{\mathrm{per}}(-L, L))$ of the regularized Boussinesq equation with any $\epsilon > 0$ and $\gamma \geq 0$.

Fixed-point iterations

The evolution problem can be written in the operator form:

$$U_{tt} - L_\epsilon U_{xx} + \epsilon\gamma L_\epsilon U = M_\epsilon U^2,$$

where

$$L_\epsilon := (1 - \epsilon\partial_x^2)^{-1}, \quad M_\epsilon := \frac{1}{2}\epsilon\partial_x^2 L_\epsilon.$$

Using Duhamel's principle, the Cauchy problem is written in the integral form:

$$U(t) = S_t(t) \star F + S(t) \star V + \int_0^t S(t-\tau) \star \left(M_\epsilon U^2(\tau) - \epsilon\gamma L_\epsilon U(\tau)\right) d\tau,$$

where the star denotes the convolution operator and $S(t)$ is the fundamental solution operator with the Fourier image:

$$\hat{S}(t) = \frac{\sin(k\hat{\ell}(k)t)}{k\hat{\ell}(k)}, \quad \ell(k) := \frac{1}{\sqrt{1+\epsilon k^2}}.$$

Fixed-point iterations

Because all operators are bounded from $L^2_{\rm per}(-L,L)$ to $L^2_{\rm per}(-L,L)$ for any $t \in \mathbb{R}$, the fixed-point iteration method yields a unique local solution in the class

$$U(t) \in C([0,t_0], H^s_{\rm per}(-L,L))$$

for any $(F,V) \in H^s_{\rm per}(-L,L) \times H^s_{\rm per}(-L,L)$ and any $s > \frac{1}{2}$, where $t_0 > 0$ is an ϵ-independent local existence time.

From the derivative equation:

$$U_t(t) = S_{tt}(t) \star F + S_t(t) \star V + \int_0^t S_t(t-\tau) \star \left(M_\epsilon U^2(\tau) - \epsilon \gamma L_\epsilon U(\tau) \right) d\tau,$$

we also obtain $U_t(t) \in C([0,t_0], H^s_{\rm per}(-L,L))$ for any $\epsilon > 0$.

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we also obtain $U_t(t) \in C([0,t_0], H^s_{\rm per}(-L,L))$ for any $\epsilon > 0$.

Remark

Note that

$$\|S_{tt}(t) \star F\|_{L^2_{\rm per}} \leq \frac{1}{\sqrt{\epsilon}} \|F\|_{L^2_{\rm per}}, \quad \epsilon > 0,$$

hence $\|U_t\|_{L^2_{\rm per}}$ *may diverge as* $\epsilon \to 0$.

Lemma

Let $U(t) \in C^1([0,t_0], H^1_{\mathrm{per}}(-L,L))$ be a local solution. The solution is extended to the time interval $[0,t_0']$ with $t_0' > t_0$ if

$$M := \sup_{t\in[0,t_0']} \|U(t)\|_{L^\infty_{\mathrm{per}}} + \sup_{t\in[0,t_0']} \|U_t(t)\|_{L^\infty_{\mathrm{per}}} < \infty.$$

Let us define the energy function

$$E(U) := \int_{-L}^{L} \left(U_t^2 + U_x^2 + \epsilon\gamma U^2 + \epsilon U_{tx}^2 + \epsilon U U_x^2\right) dx,$$

for any local solution $U(t) \in C^1([0,t_0], H^1_{\mathrm{per}}(-L,L))$. Then, we obtain

$$\frac{dE(U)}{dt} = \epsilon \int_{-L}^{L} U_t U_x^2 dx \le \epsilon \|U_t\|_{L^\infty_{\mathrm{per}}} \|U_x\|^2_{L^2_{\mathrm{per}}}.$$

Under the condition

$$M := \sup_{t\in[0,t_0']} \|U(t)\|_{L^\infty_{\text{per}}} + \sup_{t\in[0,t_0']} \|U_t(t)\|_{L^\infty_{\text{per}}} < \infty,$$

there is an ϵM-dependent constant $C(M) > 0$ such that

$$\|U_x\|^2_{L^2_{\text{per}}} \leq C(\epsilon M)E(U).$$

By Gronwall's inequality, we then obtain

$$E(U) \leq E(U_0)e^{\epsilon M C(\epsilon M)t}, \quad t \in [0, t_0'],$$

hence the solution is extended to the time $t_0' > t_0$ in the class $U(t) \in C^1([0, t_0'], H^1_{\text{per}}(-L, L))$.

Continuation criterion of local solutions with energy method

Under the condition

$$M := \sup_{t \in [0, t_0']} \|U(t)\|_{L^\infty_{\mathrm{per}}} + \sup_{t \in [0, t_0']} \|U_t(t)\|_{L^\infty_{\mathrm{per}}} < \infty,$$

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Remark

By Sobolev embedding of $H^1_{\mathrm{per}}(-L, L)$ to $L^\infty_{\mathrm{per}}(-L, L)$, we have $M = \mathcal{O}(\epsilon^{-1/2})$ and $C(\epsilon M) = \mathcal{O}(1)$ as $\epsilon \to 0$. Hence, the local solution can be continued to the time intervals of $t_0 = \mathcal{O}(\epsilon^{-1/2})$.

For smooth solutions, we have the balance equation

$$\frac{d^2}{dt^2}\int_{-L}^{L} U(x,t)dx = -\epsilon\gamma \int_{-L}^{L} U(x,t)dx,$$

hence

$$\langle U\rangle(t) := \frac{1}{2L}\int_{-L}^{L} U(x,t)dx = F_0\cos(\sqrt{\epsilon\gamma}t) + V_0\frac{\sin(\sqrt{\epsilon\gamma}t)}{\sqrt{\epsilon\gamma}}.$$

- If $\gamma = 0$, then $V_0 = 0$ to eliminate the linear growth of $\langle U\rangle$ in t.
- If $\gamma > 0$, then $\langle U\rangle$ is oscillating with the frequency $\omega = (\epsilon\gamma)^{1/2}$ in t but it diverges as $\mathcal{O}(\epsilon^{-1/2})$ unless $V_0 = 0$.

Therefore, in both cases, we impose $V_0 = 0$.

Regularized Boussinesq equation

We start with the regularized Boussinesq equation for $\gamma = 0$:

$$U_{tt} - U_{xx} = \epsilon \left(\frac{1}{2}(U^2)_{xx} + U_{ttxx} \right),$$

Note we have $\langle U \rangle = F_0$. Substituting $U(x,t) = F_0 + \tilde{U}(x,t)$, we obtain

$$\tilde{U}_{tt} - \tilde{U}_{xx} = \epsilon \left(F_0 \tilde{U}_{xx} + \frac{1}{2}(\tilde{U}^2)_{xx} + \tilde{U}_{ttxx} \right).$$

Formal asymptotic solution:

$$\tilde{U}(x,t) = U_0(x,t) + \epsilon U_1(x,t) + \epsilon^2 U_2(x,t) + \mathcal{O}(\epsilon^3).$$

In the formal theory, we collect together terms at each order. Then, we justify the approximation error by analysis of the residual equation.

Leading-order and first-order correction terms

Order $\mathcal{O}(\epsilon^0)$: The leading order U_0 satisfies the wave equation and is represented by the d'Alembert solution

$$U_0(x,t) = f^-(\xi_-, T) + f^+(\xi_+, T), \quad \xi_\pm = x \pm t, \quad T = \epsilon t,$$

where

$$f^\pm(\xi_\pm, 0) = \frac{1}{2}F(\xi_\pm) \pm \frac{1}{2}\partial_{\xi_\pm}^{-1} V.$$

Order $\mathcal{O}(\epsilon)$: The first-order correction term satisfies the inhomogeneous linear equation

$$\begin{cases} (\partial_t^2 - \partial_x^2)U_1 = -2\partial_{tT}^2 U_0 + F_0 \partial_x^2 U_0 + \frac{1}{2}\partial_x^2(U_0^2) + \partial_{ttxx}^4 U_0, \\ U_1|_{t=0} = 0, \\ \partial_t U_1|_{t=0} = -\partial_T U_0|_{t=0}. \end{cases}$$

Using the Fourier series, $U_1(x,t) = \sum_{n \in \mathbb{Z}\backslash\{0\}} g_n(t) e^{\frac{i\pi n x}{L}}$, we remove resonant (linearly growing) terms in $g_n(t)$ by requiring that $f^{\pm}(\xi_{\pm}, T)$ satisfy two uncoupled KdV equations

$$\frac{\partial}{\partial \xi_{\pm}} \left(\mp 2 \frac{\partial f^{\pm}}{\partial T} + \frac{\partial^3 f^{\pm}}{\partial \xi_{\pm}^3} + F_0 \frac{\partial f^{\pm}}{\partial \xi_{\pm}} + f^{\pm} \frac{\partial f^{\pm}}{\partial \xi_{\pm}} \right) = 0.$$

By the local and global well-posedness theory for the KdV equation, there exist unique global solutions $f^{\pm} \in C(\mathbb{R}_+, H^s_{\rm per}(-L,L))$ for any $s \geq -\frac{1}{2}$.

The first-order correction term can be written in the explicit form:

$$U_1(x,t) = f_c(x,t) + \phi^-(\xi_-, T) + \phi^+(\xi_+, T),$$

where f_c is uniquely given by

$$f_c(x,t) = -\frac{1}{4} \left(2 f^+ f^- + (\partial_{\xi_+} f^+)(\partial_{\xi_-}^{-1} f^-) + (\partial_{\xi_-} f^-)(\partial_{\xi_+}^{-1} f^+) \right),$$

whereas $\phi^{\pm}(\xi_{\pm}, 0)$ are uniquely defined from the initial data for U_1.

Justification result

Theorem

- *Assume that* $(F, V) \in H^1_{\mathrm{per}}(-L, L) \times H^1_{\mathrm{per}}(-L, L)$ *subject to the zero-mean constraint on* V, $V_0 = 0$.
- *Fix* $s \geq 10$ *and let* $f^{\pm} \in C(\mathbb{R}, H^s_{\mathrm{per}}(-L, L))$ *be global solutions of the KdV equations.*
- *Let* U_0 *and* U_1 *be given by the formal theory.*

There is $\epsilon_0 > 0$ *such that for all* $\epsilon \in (0, \epsilon_0)$ *and all* ϵ*-independent* $T_0 > 0$, *there is an* ϵ*-independent constant* $C > 0$ *such that for all* $t_0 \in [0, T_0/\epsilon]$, *the local solution of the regularized Boussinesq equation satisfies*

$$\sup_{t \in [0, t_0]} \|U - F_0 - U_0 - \epsilon U_1\|_{H^1_{\mathrm{per}}} \leq C\epsilon^2 t_0.$$

Justification analysis

Substituting

$$U(x,t) = F_0 + U_0(x,t) + \epsilon U_1(x,t) + \epsilon^2 \hat{U}(x,t),$$

we obtain the residual equation for the error term

$$\hat{U}_{tt} - (1+\epsilon F_0)\hat{U}_{xx} - \epsilon \hat{U}_{ttxx} = \epsilon \partial_x^2 \left(U_0 \hat{U} + \epsilon U_1 \hat{U} + \frac{1}{2}\epsilon^2 \hat{U}^2 \right) + \hat{H},$$

where $\hat{H} = -2\partial_t \partial_T U_1 - \partial_T^2 U_0 - \epsilon \partial_T^2 U_1 + ...$.

There exists a unique solution $\hat{U} \in C^1([0,t_0], H^1_{\rm per}(-L,L))$ if $\hat{H} \in C([0,t_0], H^1_{\rm per}(-L,L))$. This require global solutions of the KdV equations $f^{\pm} \in C(\mathbb{R}, H^s_{\rm per}(-L,L))$ to exist for $s \geq 10$.

Using the energy

$$\hat{E} = \int_{-L}^{L} \left(\hat{U}_t^2 + (1 + \epsilon F_0)\hat{U}_x^2 + \epsilon \hat{U}_{tx}^2 + \epsilon U_0 \hat{U}_x^2 + 2\epsilon U_{0x} \hat{U} \hat{U}_x \right) dx,$$

and Poincaré's inequality for $(2L)$-periodic mean-zero functions,

$$\|\hat{U}\|_{L^2_{\mathrm{per}}}^2 \leq C \|\hat{U}_x\|_{L^2_{\mathrm{per}}}^2 \leq C\hat{E},$$

we obtain the a priori energy inequality

$$\frac{1}{2}\frac{d\hat{E}}{dt} \leq \|\hat{H}\|_{L^2_{\mathrm{per}}} \|\hat{U}_t\|_{L^2_{\mathrm{per}}} + C\epsilon \left(\|U_0\|_{L^\infty_{\mathrm{per}}} + \epsilon^{1/2} \|U_1\|_{L^\infty_{\mathrm{per}}} + \epsilon^{3/2} \|\hat{U}\|_{L^\infty_{\mathrm{per}}} \right) \hat{E},$$

Setting $\hat{E} := \hat{Q}^2$ and using Sobolev embedding's $\|\hat{U}\|_{L^\infty_{\mathrm{per}}} \leq C_{\mathrm{emb}} \hat{Q}$, we write

$$\frac{d\hat{Q}}{dt} \leq \|\hat{H}\|_{L^2_{\mathrm{per}}} + C\epsilon \left(\|U_0\|_{L^\infty_{\mathrm{per}}} + \epsilon^{1/2} \|U_1\|_{L^\infty_{\mathrm{per}}} + \epsilon^{3/2} \hat{Q} \right) \hat{Q}.$$

By Gronwall's inequality, we have

$$\hat{Q}(t) \leq \left(\hat{Q}(0) + t_0 \sup_{t \in [0,t_0]} \|\hat{H}\|_{L^2_{\rm per}} \right) e^{C\epsilon t}, \quad t \in [0, t_0],$$

for any $t_0 > 0$, sufficiently small ϵ, and some (t_0, ϵ)-independent positive constant C_0.

Since $\hat{Q}(0) = \|\partial_T U_1\|_{L^2_{\rm per}}$, this estimate yields the bound

$$\sup_{t \in [0,t_0]} \|U - F_0 - U_0 - \epsilon U_1\|_{H^1_{\rm per}} \leq C\epsilon^2 t_0.$$

The smooth solution is continued from $t_0 = \mathcal{O}(1)$ to $t_0 = \mathcal{O}(\epsilon^{-1})$ thanks to $\hat{Q}(t) < \infty$ for all $t \in [0, t_0]$, as well as to Sobolev's embeddings

$$\|\epsilon^2 \hat{U}\|_{L^\infty_{\rm per}} \leq C_{\rm emb} \epsilon^2 \hat{Q}, \quad \|\epsilon^2 \hat{U}_t\|_{L^\infty_{\rm per}} \leq C_{\rm emb} \epsilon^{3/2} \hat{Q}.$$

Second-order correction terms

Order $\mathcal{O}(\epsilon^2)$: The second-order correction term satisfies the linear inhomogeneous equation

$$\begin{cases} (\partial_t^2 - \partial_x^2)U_2 = -2\partial_{tT}^2 U_1 - \partial_T^2 U_0 + c_0 \partial_x^2 U_1 + \partial_x^2 (U_0 U_1) + \partial_{ttxx}^4 U_1 + 2\partial_{tTxx}^4 U_0, \\ U_2|_{t=0} = 0, \\ \partial_t U_2|_{t=0} = -\partial_T U_1|_{t=0}. \end{cases}$$

Using the Fourier series again $U_2(x,t) = \sum_{n \in \mathbb{Z}\setminus\{0\}} g_n(t) e^{\frac{i\pi n x}{L}}$, we eliminate the linearly growth in t and obtain the linearized KdV equations for $\phi^\pm(\xi_\pm, T)$:

$$\frac{\partial}{\partial \xi_\pm}\left(\mp 2\frac{\partial \phi^\pm}{\partial T} + \frac{\partial^3 \phi^\pm}{\partial \xi_\pm^3} + c_0 \frac{\partial \phi^\pm}{\partial \xi_\pm} + \frac{\partial}{\partial \xi_\pm} f^\pm \phi^\pm\right) = \frac{\partial^2 f^\pm}{\partial T^2} \mp 2\frac{\partial^4 f^\pm}{\partial \xi_\pm^3 T} + \frac{\partial^2 f_s^\pm}{\partial \xi_\pm^2},$$

where

$$f_s^\pm(\xi_\pm, T) = \frac{1}{4L} f^\pm(\xi_\pm, T) \int_{-L}^{L} |f^\mp(\xi, T)|^2 d\xi.$$

Then, there exists a bounded solution for U_2 in t.

Justification result

Theorem

- *Assume that* $(F, V) \in H^1_{\mathrm{per}}(-L, L) \times H^1_{\mathrm{per}}(-L, L)$ *subject to the zero-mean constraint on* V*,* $V_0 = 0$*.*
- *Fix* s *sufficiently large and let* $f^{\pm} \in C(\mathbb{R}, H^s_{\mathrm{per}}(-L, L))$ *be global solutions of the KdV equations.*
- *Let* U_0 *and* U_1 *be given by the formal theory.*
- *Let* $\phi^{\pm}$ *satisfy the linearized KdV equations.*

Then, for all $\epsilon \in (0, \epsilon_0)$ *and all* ϵ*-independent* $T_0 > 0$*, there is an* ϵ*-independent constant* $C > 0$ *such that*

$$\sup_{t \in [0, T_0/\epsilon]} \|U - c_0 - U_0 - \epsilon U_1\|_{H^1_{\mathrm{per}}} \leq C\epsilon^2.$$

The proof is similar and is based on the decomposition

$$U(x,t) = F_0 + U_0(x,t) + \epsilon U_1(x,t) + \epsilon^2 U_2(x,t) + \epsilon^3 \hat{U}(x,t).$$

Numerical example

Consider the initial data

$$\begin{cases} U|_{t=0} = 3k^2 \operatorname{sech}^2\left(\frac{kx}{2}\right), \\ U_t|_{t=0} = 3k^3 \operatorname{sech}^2\left(\frac{kx}{2}\right) \tanh\left(\frac{kx}{2}\right), \end{cases}$$

where $k > 0$ is an arbitrary parameter. The initial data is defined on the periodic domain $-L \leq x \leq L$ and the mean value is given by

$$F_0 = \frac{1}{2L} \int_{-L}^{L} U|_{t=0} dx = \frac{6k}{L} \tanh\left(\frac{kL}{2}\right).$$

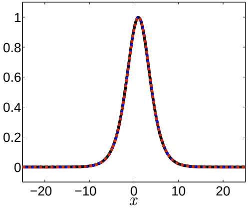
When $L \to \infty$, $F_0 \to 0$, the initial data corresponds to a solitary wave of the KdV equation propagating to the right.

Weakly nonlinear approximations:

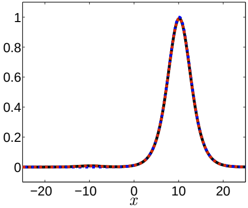
$$U_1 = f^-, \quad U_{12} = f^- + \epsilon\left(\phi_-|_{T=0} + \phi_+|_{T=0}\right), \quad U_2 = f^- + \epsilon\left(\phi_- + \phi_+\right).$$

Comparison of the numerical solutions: $\epsilon = 0.1$

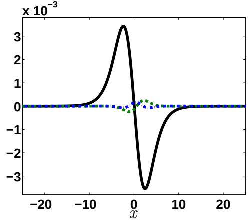
$t = 1$ $\qquad$ $t = 1/\epsilon$



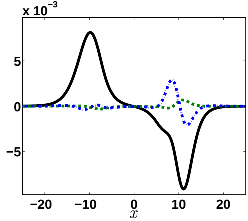
(a) U_{num}(—) U_1(-.-) U_{12}(· · ·) U_2 (- -)



(b) U_{num}(—) U_1(-.-) U_{12}(· · ·) U_2 (- -)



(c) $(U_{\mathrm{num}} - U_1)$(—) $(U_{\mathrm{num}} - U_{12})$(-.-) $(U_{\mathrm{num}} - U_2)$(- -)



(d) $(U_{\mathrm{num}} - U_1)$(—) $(U_{\mathrm{num}} - U_{12})$(-.-) $(U_{\mathrm{num}} - U_2)$(- -)

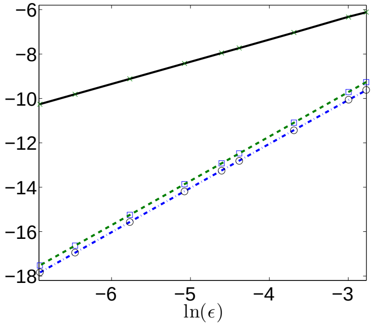
Theoretical results

$$\sup_{t\in[0,t_0]} \|U - F_0 - U_0 - \epsilon U_1|_{T=0}\|_{H^1_{\mathrm{per}}} \leq C\epsilon^2 t_0$$

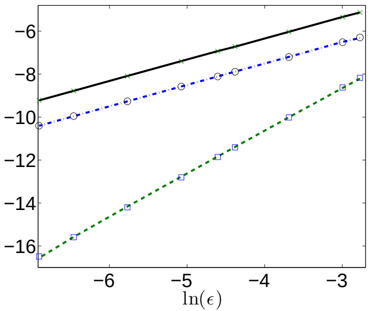
and

$$\sup_{t\in[0,t_0]} \|U - F_0 - U_0 - \epsilon U_1\|_{H^1_{\mathrm{per}}} \leq C\epsilon^2.$$

$t = 1$



$t = 1/\epsilon$



Regularized Boussinesq–Ostrovsky equation

We finish with the regularized Boussinesq-Ostrovsky equation for $\gamma > 0$:

$$U_{tt} - U_{xx} = \epsilon \left(\frac{1}{2}(U^2)_{xx} + U_{ttxx} - \gamma U \right).$$

Now we have $\langle U \rangle = F_0 \cos(\omega t)$ with $\omega = \sqrt{\epsilon\gamma}$. Substituting $U(x,t) = F_0 \cos(\omega t) + \tilde{U}(x,t)$, we obtain

$$\tilde{U}_{tt} - \tilde{U}_{xx} = \epsilon \left(F_0 \cos(\omega t) \tilde{U}_{xx} + \frac{1}{2}(\tilde{U}^2)_{xx} + \tilde{U}_{ttxx} - \gamma \tilde{U} \right).$$

Formal asymptotic solution:

$$\tilde{U}(x,t) = U_0(x,t) + \epsilon U_1(x,t) + \epsilon^2 U_2(x,t) + \mathcal{O}(\epsilon^3).$$

In the formal theory, we collect together terms at each order. Then, we justify the approximation error by analysis of the residual equation.

Leading-order and first-order correction terms

Order $\mathcal{O}(\epsilon^0)$: The leading order U_0 satisfies the wave equation and is represented by the d'Alembert solution

$$U_0(x,t) = f^-(\xi_-,T) + f^+(\xi_+,T), \quad \xi_\pm = x \pm t, \quad T = \epsilon t,$$

where

$$f^\pm(\xi_\pm,0) = \frac{1}{2}F(\xi_\pm) \pm \frac{1}{2}\partial_{\xi_\pm}^{-1}V.$$

Order $\mathcal{O}(\epsilon)$: The first-order correction term satisfies the inhomogeneous linear equation

$$\begin{cases} (\partial_t^2 - \partial_x^2)U_1 = -2\partial_{tT}^2 U_0 + F_0\cos(\omega t)\partial_x^2 U_0 + \frac{1}{2}\partial_x^2(U_0^2) + \partial_{ttxx}^4 U_0 - \gamma U_0, \\ U_1|_{t=0} = 0, \\ \partial_t U_1|_{t=0} = -\partial_T U_0|_{t=0}. \end{cases}$$

Using the Fourier series, $U_1(x,t) = \sum_{n\in\mathbb{Z}\backslash\{0\}} g_n(t) e^{\frac{i\pi n x}{L}}$, we remove resonant (linearly growing) terms in $g_n(t)$ by requiring that $f^{\pm}(\xi_{\pm}, T)$ satisfy two uncoupled Ostrovsky equations

$$\frac{\partial}{\partial \xi_{\pm}} \left(\mp 2 \frac{\partial f^{\pm}}{\partial T} + \frac{\partial^3 f^{\pm}}{\partial \xi_{\pm}^3} + f^{\pm} \frac{\partial f^{\pm}}{\partial \xi_{\pm}} \right) = \gamma f^{\pm}.$$

Note that the oscillating F_0 term does not enter the Ostrovsky equation and that the mean-zero constraint is satisfied automatically.

The first-order correction term is found from the inhomogeneous equations

$$\begin{aligned} \frac{d^2 g_n}{dt^2} + \left(\frac{\pi n}{L}\right)^2 g_n \quad &= \quad -\frac{\pi^2 n^2}{L^2} F_0 \cos(\omega t) \left(a_n^+ e^{\frac{i\pi n t}{L}} + a_n^- e^{-\frac{i\pi n t}{L}} \right) \\ &\qquad - \frac{\pi^2 n^2}{L^2} \sum_{k\in\mathbb{Z}\backslash\{0,n\}} a_k^+ a_{n-k}^- e^{\frac{i\pi(2k-n)t}{L}}, \end{aligned}$$

As a result, we have

$$\|\epsilon U_1\|_{H^1_{\rm per}} = \mathcal{O}(F_0 \epsilon^{1/2}) \quad \text{as} \quad \epsilon \to 0.$$

Justification result

Theorem

- *Assume that* $(F, V) \in H^1_{\mathrm{per}}(-L, L) \times H^1_{\mathrm{per}}(-L, L)$ *subject to the zero-mean constraint on* V, $V_0 = 0$.
- *Fix* $s \geq 10$ *and let* $f^{\pm} \in C(\mathbb{R}, H^s_{\mathrm{per}}(-L, L))$ *be global solutions of the Ostrovsky equations.*
- *Let* U_0 *and* U_1 *be given by the formal theory.*

There is $\epsilon_0 > 0$ *such that for all* $\epsilon \in (0, \epsilon_0)$ *and all* ϵ*-independent* $T_0 > 0$, *there is an* ϵ*-independent constant* $C > 0$ *such that for all* $t_0 \in [0, T_0/\epsilon]$, *the local solution of the regularized Boussinesq–Ostrovsky equation satisfies*

$$\sup_{t \in [0, t_0]} \|U - F_0 \cos(\omega t) - U_0 - \epsilon U_1\|_{H^1_{\mathrm{per}}} \leq C \epsilon t_0 (F_0 + \epsilon).$$

If, in addition, $\phi^{\pm}$ *in* U_1 *satisfy the linearized Ostrovsky equation, then*

$$\sup_{t \in [0, T_0/\epsilon]} \|U - F_0 \cos(\omega t) - U_0 - \epsilon U_1\|_{H^1_{\mathrm{per}}} \leq C \epsilon (F_0 + \epsilon).$$

Numerical example

Consider the initial data

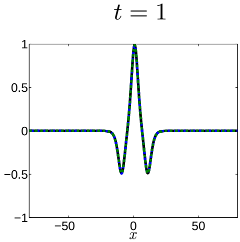
$$\begin{cases} U|_{t=0} = 3k^2 \operatorname{sech}^2(\frac{kx}{2}) - \hat{\alpha}[\operatorname{sech}^2\left(\frac{k(x+x_0)}{2}\right) + \operatorname{sech}^2\left(\frac{k(x-x_0)}{2}\right)], \\ U_t|_{t=0} = 3k^3 \operatorname{sech}^2(\frac{kx}{2}) \tanh(\frac{kx}{2}) - k\hat{\alpha}\Big[\operatorname{sech}^2\left(\frac{k(x+x_0)}{2}\right) \tanh\left(\frac{k(x+x_0)}{2}\right) \\ \qquad\qquad + \operatorname{sech}^2\left(\frac{k(x-x_0)}{2}\right) \tanh\left(\frac{k(x-x_0)}{2}\right)\Big], \end{cases}$$

where $x_0 > 0$ is an arbitrary shift and $\hat{\alpha}$ is chosen to satisfy $F_0 = 0$.

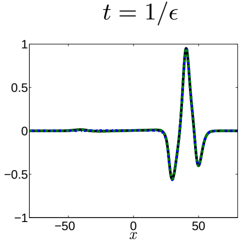
Weakly nonlinear approximations:

$$U_1 = f^-, \quad U_2 = f^- + \epsilon\left(\phi_- + \phi_+\right).$$

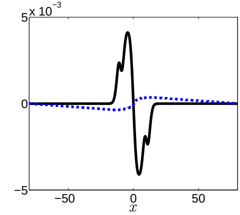
Comparison of the numerical solutions: $\epsilon = 0.025$, $\gamma = 0.1$



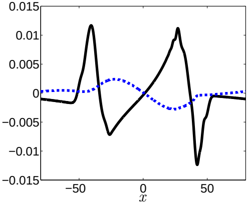
(a) U_{num}**(—)** U_1**(-.-)** U_2**(- -)**



(b) U_{num}**(—)** U_1**(-.-)** U_2**(- -)**



(c) $(U_{\mathrm{num}} - U_1)$**(—)** $(U_{\mathrm{num}} - U_2)$**(- -)**



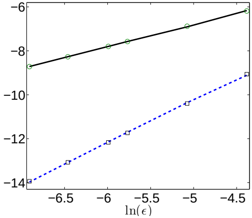
(d) $(U_{\mathrm{num}} - U_1)$**(—)** $(U_{\mathrm{num}} - U_2)$**(- -)**

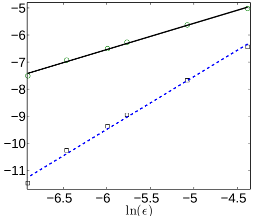
Convergence of the approximation errors in ϵ

Theoretical result

$$\sup_{t\in[0,t_0]} \|U - U_0 - \epsilon U_1\|_{H^1_{\mathrm{per}}} \leq C\epsilon^2.$$

$t = 1$ $\qquad\qquad$ $t = 1/\epsilon$





$\ln(e_t^1)$(—) $\quad \alpha_1 \ln(\epsilon) + \ln(C_1)$(○ ○) $\quad \ln(e_t^2)$(- -) $\quad \alpha_2 \ln(\epsilon) + \ln(C_2)$(□ □)

Conclusion

- The presence of the dispersion with γ term introduces oscillations of the mean term and deteriorates convergence of the approximation errors with ϵ.
- Mean-zero terms are naturally incorporated in the reduced amplitude equations and in the approximation analysis.
- Natural energy H^1 spaces are incorporated in local well-posedness, solution continuation, and a priori energy estimate analysis.
- Theoretical results are fully confirmed with numerical approximations.