Multi-component vortices in coupled NLS equations

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Reference: Fundamentalnaya i prikladnaya matematika **12**, 35–63 (2006)

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Scalar vortices

Main model: Focusing two-dimensional NLS equation

$$iu_t + u_{xx} + u_{yy} + f(|u|^2)u = 0,$$

where $f(s)$ is $C^1(\mathbb{R}_+)$ and $f'(s) > 0$ on $s \in \mathbb{R}_+$.

Definition: Vortices are stationary solutions of the form

$$u = R(r)e^{im\theta}e^{i\omega t}$$

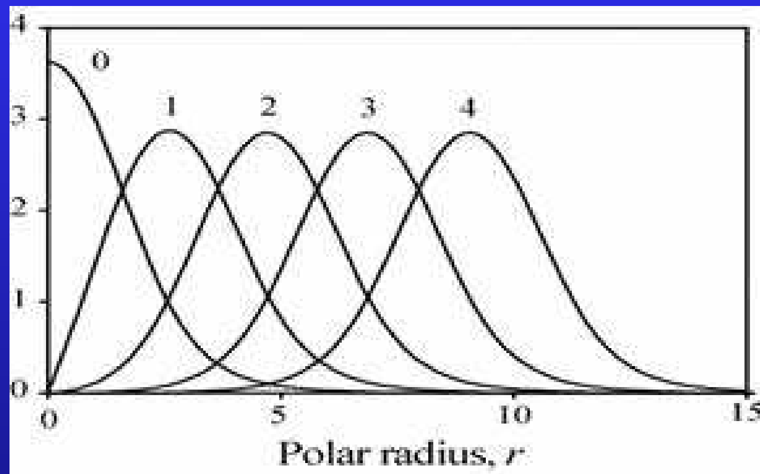
where (r, θ) are polar coordinates on $\mathbb{R}^2$, $m \in \mathbb{N}$ is the vortex charge and $R(r)$ is a solution of the second-order ODE:

$$R'' + \frac{1}{r}R' - \frac{m^2}{r^2}R - \omega R + f(R^2)R = 0,$$

such that $R(r) \to r^{|m|}$ as $r \to 0$ and $R(r) \to e^{-\sqrt{\omega}r}/\sqrt{r}$ as $r \to \infty$.

Examples

Saturable medium with $f(s) = s/(1+s)$

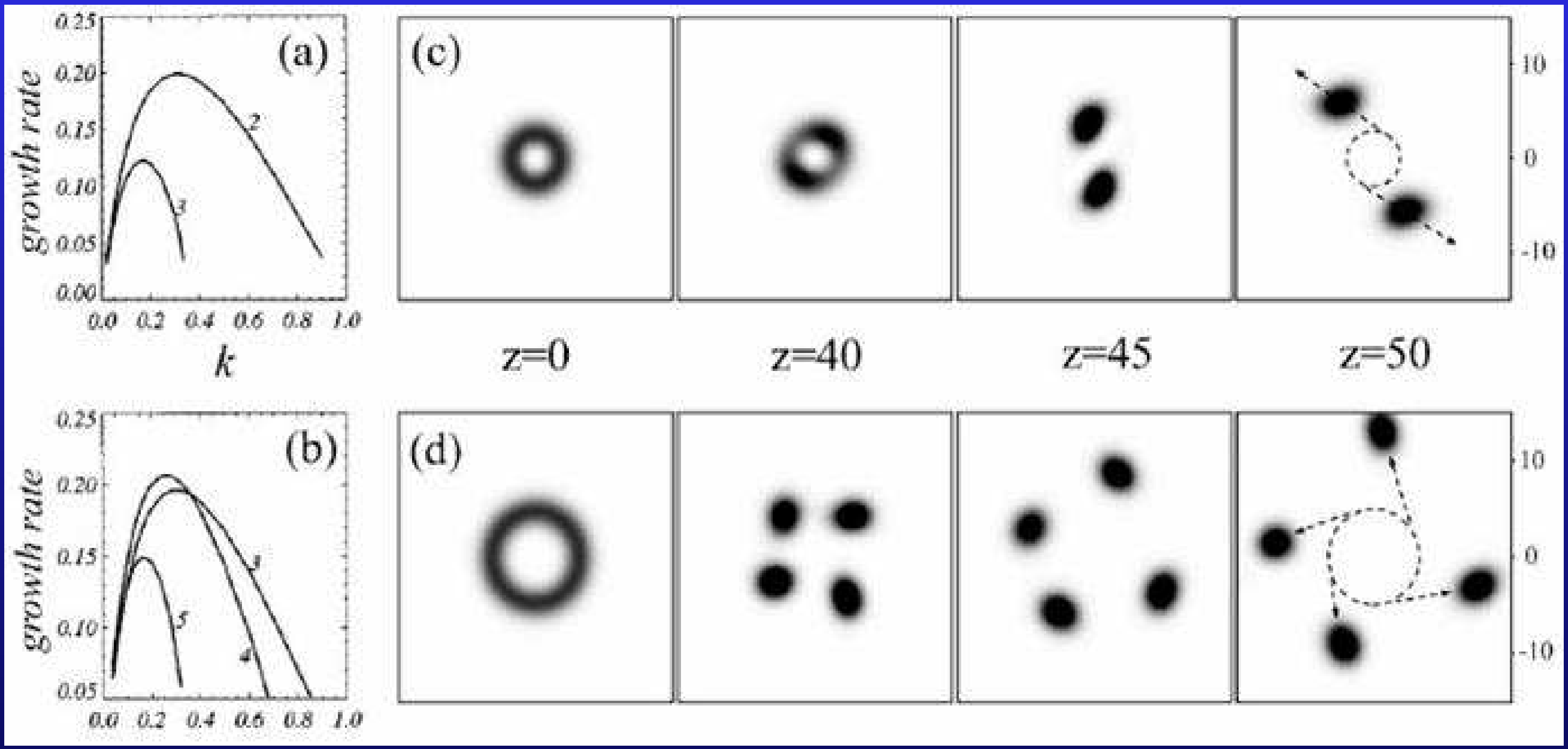


Cubic-quintic approximation $f(s) = s - s^2$

Desyatnikov, Kivshar, and Torner, Optical Vortices and Vortex Solitons, In: Progress in Optics 47, Ed. E. Wolf (2005)

Typical instabilities

Vortices with charges $m = 1$ and $m = 2$ are unstable in the cubic NLS model:



Skryabin and Firth, Phys. Rev. E 58, 3916 (1998)

Coupled NLS equations

We consider the system of n coupled NLS equations

$$i\dot{\psi}_k + \Delta\psi_k = W'(E)\psi_k, \qquad k = 1, 2, ..., n,$$

where $\psi_k : \mathbb{R}^2 \times \mathbb{R} \mapsto \mathbb{C}$ and $W : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is C^2-function of $E = \sum_{k=1}^n |\psi_k|^2$. The system is Hamiltonian with

$$H = \int_{\mathbb{R}^2} \left[\sum_{k=1}^n \left(|\partial_x \psi_k|^2 + |\partial_y \psi_k|^2 \right) + W\left(\sum_{k=1}^n |\psi_k|^2 \right) \right] dxdy$$

Vector momentum, angular momentum, and the charges

$$Q_k = \int_{\mathbb{R}^2} |\psi_k|^2 dxdy, \qquad k = 1, ..., n$$

are *basic* conserved quantities of the system.

Symplectic symmetries

The Hamiltonian function is invariant with respect to N-parameter group of rotations in the space $\mathbb{R}^{2n}$ or $\mathbb{C}^n \times \bar{\mathbb{C}}^n$, where

$$N = \begin{pmatrix} 2n \\ 2 \end{pmatrix} = \frac{(2n)!}{2!(2n-2)!} = n(2n-1).$$

Let G be a linear transformation in $x \in \mathbb{R}^{2n}$ and J be the symplectic matrix such that $\dot{x} = J\mathrm{grad}(H(x))$. Then, the Hamiltonian system is invariant under the linear transformation if and only if

$$J = GJG^T$$

Such transformations G are called *symplectic transformations*. If $G = I + g$, then $gJ + Jg^T = 0$.

Symmetries of the coupled NLS

Lemma: The group of symplectic rotations in $\mathbb{R}^{2n}$ is generated by the n^2-parameter matrix

$$g = \begin{pmatrix} A & B \\ -B^T & A \end{pmatrix},$$

where $A^T = -A$ and $B^T = B$. Equivalently, the group of symplectic rotations in $\mathbb{C}^n$ is generated by $g = A - iB$.

Additional conserved quantities related to the symmetries:

$$Q_{k,m} = \int_{\mathbb{R}^2} \psi_k \bar{\psi}_m dx dy, \qquad k = 1, ..., n, \quad m = k, ..., n,$$

where n quantities are real-valued charges $Q_k = Q_{k,k}$ and $n(n-1)/2$ quantities $Q_{k,m}$ with $m \neq k$ are complex-valued.

Example $n = 2$

Symplectic rotations in $\mathbb{C}^2$

$$G_1 = \begin{pmatrix} e^{i\theta_1} & 0 \\ 0 & 1 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta_2} \end{pmatrix}$$

$$G_3 = \begin{pmatrix} \cos\theta_3 & \sin\theta_3 \\ -\sin\theta_3 & \cos\theta_3 \end{pmatrix}, \qquad G_4 = \begin{pmatrix} \cos\theta_4 & i\sin\theta_4 \\ i\sin\theta_4 & \cos\theta_4 \end{pmatrix},$$

where $\theta_{1,2,3,4}$ are defined on $[0, 2\pi]$. If (ψ_1, ψ_2) is a solution of the coupled NLS equations, then a new solution $(\tilde{\psi}_1, \tilde{\psi}_2)$ is

$$\tilde{\psi}_1 = \alpha_1 e^{i\theta_1}\psi_1 + \alpha_2 e^{i\theta_2}\psi_2,$$
$$\tilde{\psi}_2 = -\bar{\alpha}_2 e^{i\theta_1}\psi_1 + \bar{\alpha}_1 e^{i\theta_2}\psi_2,$$

where (α_1, α_2) are complex-valued parameters and (θ_1, θ_2) are real-valued parameters.

Stationary solutions : classification

Consider the stationary solution in the form:

$$\psi_k = \varphi_k(x)e^{i\omega_k t}, \qquad k = 1, ..., n,$$

where ω_k is real-valued parameter and $\varphi_k(x)$ solves

$$\Delta\varphi_k - \omega_k\varphi_k = W'(E)\varphi_k$$

First Alternative:

For any $k \neq m$, either (I) $\omega_k = \omega_m$ or (II) $(\varphi_k, \varphi_m) = 0$.

Class I contains a variety of super-symmetric vortex congifurations.
Class II contains a variety of coupled states between solitons and vortices of different frequencies.

Vortex solutions of class I

Separating the polar coordinates (r, θ) in $\mathbb{R}^2$:

$$\varphi_k = \phi_k(\theta) R_k(r), \qquad k = 1, ..., n,$$

we obtain two ODEs:

$$\phi_k'' + m_k^2 \phi_k = 0, \qquad R_k'' + \frac{1}{r} R_k' - \frac{m_k^2}{r^2} R_k = (W'(E_0) + \omega) R_k,$$

where $E_0(r) = \sum_{k=1}^n R_k^2(r) |\phi_k(\theta)|^2$. By the periodicity of $\phi_k(\theta)$, parameter m_k is integer (*vortex charge*).

Second Alternative:

For any $k \neq m$, either (i) $m_k^2 = m_m^2$, $R_k = R_m$ or (ii) $\int_0^\infty R_k(r) R_m(r) dr/r = 0$, $\phi_k(\theta) = e^{\pm i m_k \theta}$.

Vortex solutions of sub-class I (i)

Let $m_k^2 = m^2$ and $R_k = R(r)\ \forall k$, where

$$R'' + \frac{1}{r}R' - \frac{m^2}{r^2}R = (W'(R^2) + \omega)R.$$

Then,

$$\phi_k = a_k e^{im\theta} + b_k e^{-im\theta}, \qquad k = 1, ..., n,$$

subject to the solvability constraint $\sum_{k=1}^{n} |\phi_k|^2 = 1$, which is equivalent to the normalization in $\mathbb{C}^n$:

$$\|\mathbf{a}\|^2 + \|\mathbf{b}\|^2 = 1, \qquad (\mathbf{a}, \mathbf{b}) = 0,$$

where $\mathbf{a} = (a_1, ..., a_n)^T$ and $\mathbf{b} = (b_1, ..., b_n)^T$ are in $\mathbb{C}^n$.

Examples for $n = 2$

- If $\mathbf{b} = \mathbf{0}$, then the solution is *vortex pair of double charge*:

$$\varphi_1 = a_1 R(r) e^{im\theta}, \qquad \varphi_2 = a_2 R(r) e^{im\theta},$$

  which is equivalent to the scalar vortex $\mathbf{a} = (1,0)^T$ by symplectic rotations

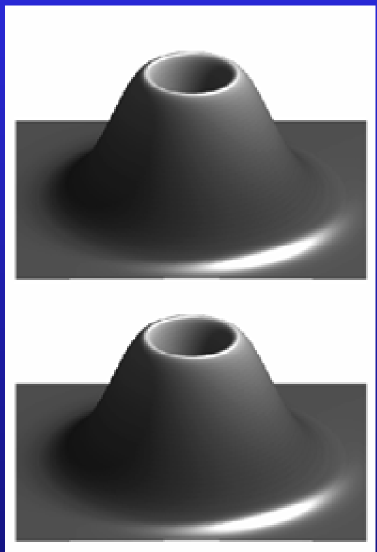
- If $\mathbf{a} = \frac{1}{\sqrt{2}}(1,0)^T$ and $\mathbf{b} = \frac{1}{\sqrt{2}}(0,1)^T$, then the solution is *vortex pair of hidden charge*:

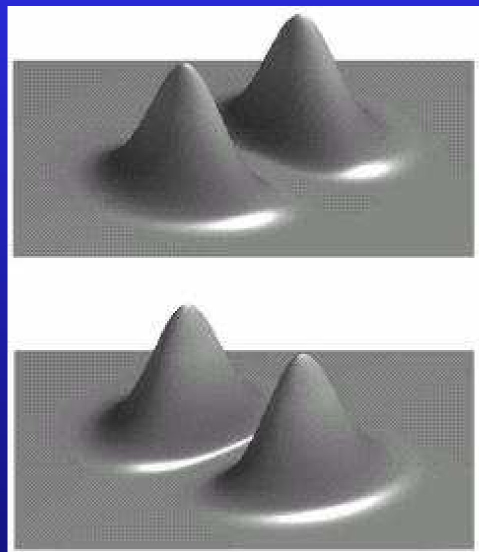
$$\varphi_1 = \frac{1}{\sqrt{2}} R(r) e^{im\theta}, \qquad \varphi_2 = \frac{1}{\sqrt{2}} R(r) e^{-im\theta}.$$

  which generates a family with $\mathbf{a}, \mathbf{b} \neq \mathbf{0}$ by symplectic rotations

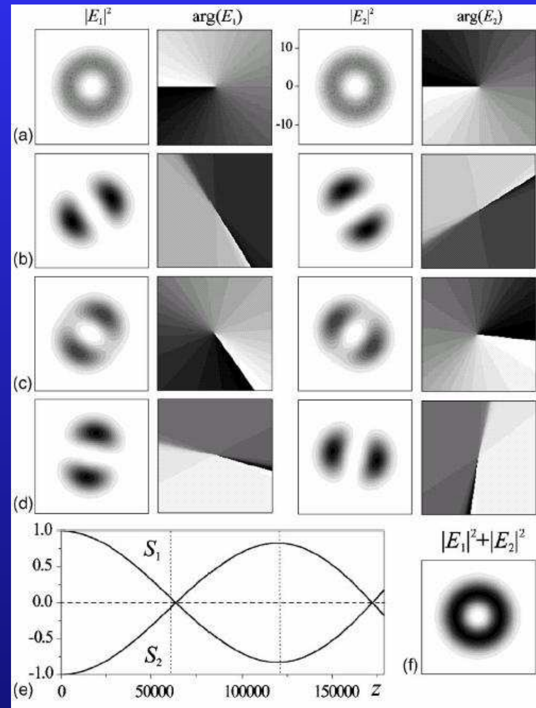
Examples for $n = 2$

Cubic model
(Manakov model)





Cubic–quntic model



Vortex solutions of sub-class I (ii)

Let $\phi_k(\theta) = e^{\pm i m_k \theta}$ with m_k^2 being distinct for all k. There exists an invariant manifold $R_k(r) \equiv 0$ for any k.

A hierarchy of vortex states:

- Scalar vortex

$$\varphi_1 = R(r) e^{im\theta}, \;\; \varphi_k = 0, \;\; k = 2, ..., n.$$

- Two-charge vortex

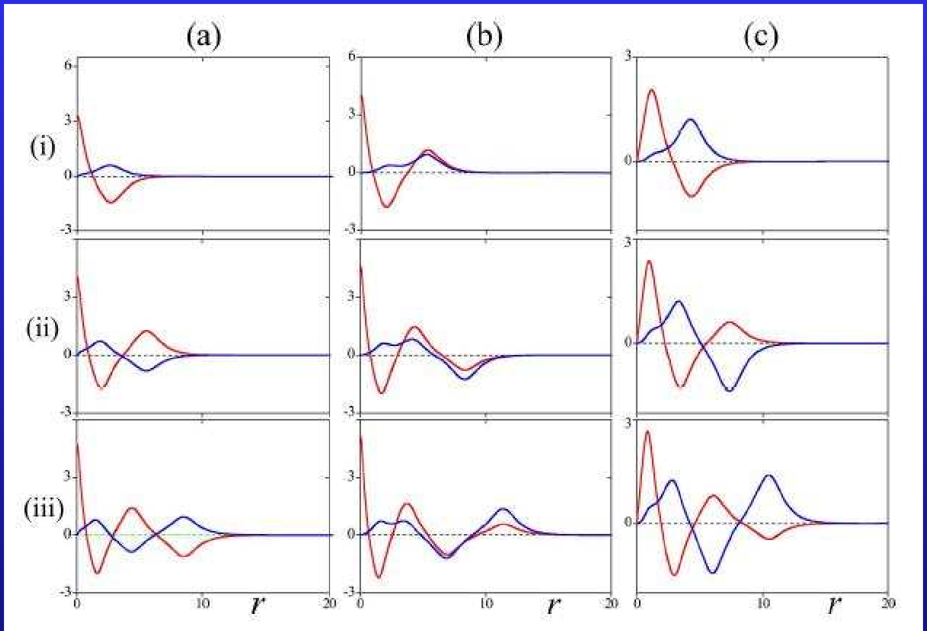
$$\varphi_1 = \alpha_1 R_1(r) e^{im_1\theta}, \;\; \varphi_2 = \alpha_2 R_2(r) e^{im_2\theta}, \;\; \varphi_k = 0, \;\; k = 3, ...n,$$

where $R_{1,2}(r)$ solve $L_{m_1} R_1 = 0$ and $L_{m_2} R_2 = 0$ with

$$L_m = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{m^2}{r^2} - \omega + W'(R_1^2 + R_2^2).$$

Examples for $n = 2$

Cubic (Manakov) system



(a) $(m_1, m_2) = (0, 1)$, (b) $(m_1, m_2) = (0, 2)$, (c) $(m_1, m_2) = (1, 2)$.

Conjecture from the numerical observations:

$$|m_1| + n_1 = |m_2| + n_2,$$

where n_k is the number of nodes of $R_k(r)$ on $r > 0$.

Symmetries and linearization

Stability problem for perturbation vector $(\mathbf{u}, \mathbf{w}) \in \mathbb{C}^{2n}$ is

$$\mathcal{H} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} = i\lambda \begin{pmatrix} I & O \\ O & -I \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix},$$

where

$$\mathcal{H} = (-\Delta + W'(E)) \begin{pmatrix} I & O \\ O & I \end{pmatrix} + W''(E) \begin{pmatrix} \boldsymbol{\varphi} \\ \bar{\boldsymbol{\varphi}} \end{pmatrix} \cdot \left(\bar{\boldsymbol{\varphi}}^T \;\; \boldsymbol{\varphi}^T \right)$$

and $E(x) = \sum_{k=1}^{n} |\varphi_k(x)|^2$.

Remark: The last term in $\mathcal{H}$ is the outer product, which is a rank-one matrix for any $x \in \mathbb{R}^2$. This can be used for block-diagonalization and simplification of the stability problem.

Example: vortices of sub-class I(ii)

For instance, consider a scalar vortex:

$$\varphi_k(x) = a_k R(r) e^{im\theta}, \qquad a_k \in \mathbb{C} : \quad \sum_{k=1}^{n} |a_k|^2 = 1$$

An ortho-normal basis in $\mathbb{C}^n$: $S_1 = \{\mathbf{a}, \mathbf{c}_1, \mathbf{c}_2, \ldots, \mathbf{c}_{n-1}\}$, where $\{\mathbf{c}_j\}_{j=1}^{n-1}$ spans the orthogonal compliment of $\mathbf{a}$. The decomposition

$$\mathbf{u}(x) = \alpha^+(x)\mathbf{a} + \sum_{j=1}^{n-1} \gamma_j^+(x)\mathbf{c}_j, \quad \mathbf{v}(x) = \alpha^-(x)\bar{\mathbf{a}} + \sum_{j=1}^{n-1} \gamma_j^-(x)\bar{\mathbf{c}}_j$$

block-diagonalizes $\mathcal{H}$ into 2-by-2 coupled non-self-adjoint problem for (α^+, α^-) and pairs of uncoupled self-adjoint problems for $\{\gamma_j^+, \gamma_j^-\}_{j=1}^{n-1}$.

Example: vortices of sub-class I(i)

For instance, consider a vortex pair with hidden charge

$$\varphi_k(x) = \left(a_k e^{im\theta} + b_k e^{-im\theta}\right) R(r), \qquad a_k, b_k \in \mathbb{C},$$

such that

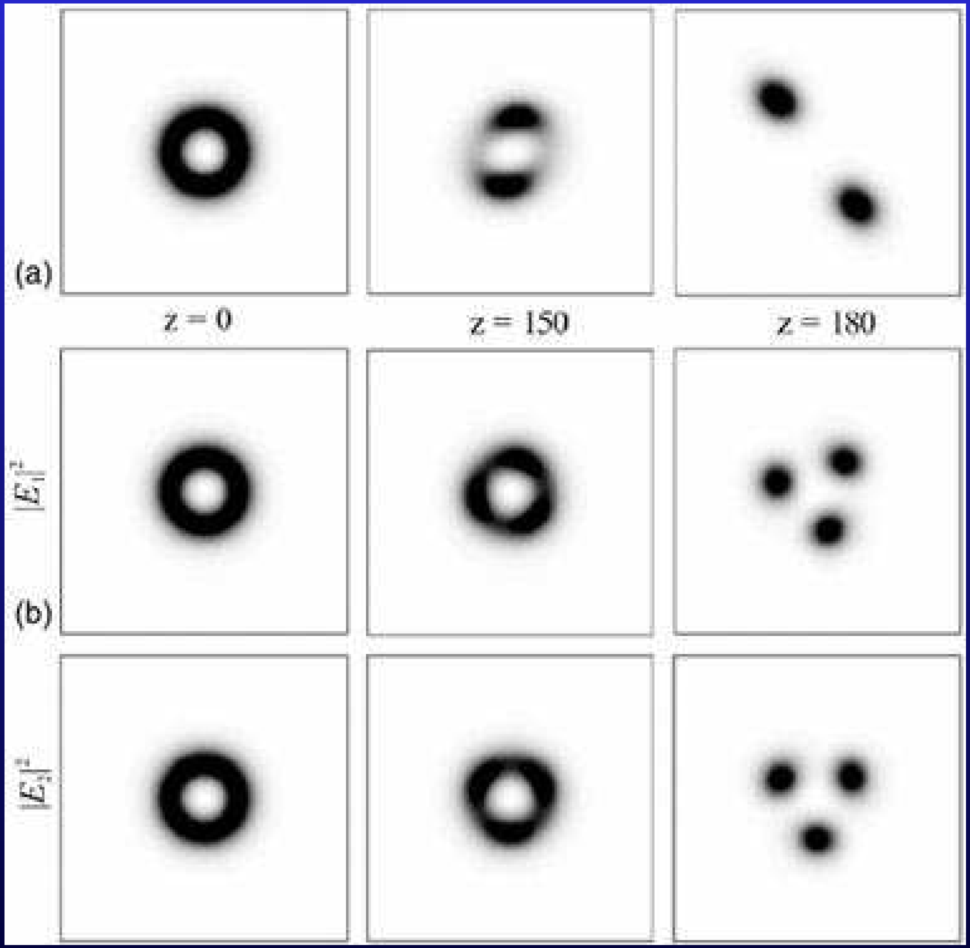
$$(\mathbf{a}, \mathbf{b}) = 0, \qquad \|\mathbf{a}\|^2 = \frac{1+\mu}{2}, \qquad \|\mathbf{b}\|^2 = \frac{1-\mu}{2}.$$

A similar decomposition over an ortho-normal basis $S_2 = \{\mathbf{a}, \mathbf{b}, \mathbf{c}_1, \ldots, \mathbf{c}_{n-2}\} \subset \mathbb{C}^n$ block-diagonalizes $\mathcal{H}$ into 4-by-4 coupled non-self-adjoint problem and pairs of uncoupled self-adjoint problems.

Example: instabilities of vortex pairs

(a) $\mu = \pm 1$ - vortex of double charge

(b) $\mu = 0$ - vortex of hidden charge



Conclusion

- Analysis of symplectic rotations and conserved quantities
- Classification and simplification of exact vortex solutions and their linearizations in the system of coupled NLS equations
- An algorithm for analysis of a particular family of solutions
  1. Construct the seed vortex for a given vortex solution
  2. Study analytically and numerically the associated linearization problem for the seed vortex
  3. Rotate the seed vortex and the eigenvectors of the linearized problem with the same group of symplectic rotations to obtain relevant results for the given vortex.
- Nonlinear dynamics of solutions along the family of symplectic rotations is not yet well understood.