Modeling of low-contrast photonic crystals with coupled-mode equations

Dmitri Agueev
Mathematics and Statistics
McMaster

Outline

- *Introduction*
- *Resonances*
- *Coupled-mode equations*
- *Analysis: existence and uniqueness*
- *Explicit solutions*

Introduction

- ***Motivation: control the optical properties of materials***
  - Fiber-optics
  - Lasers
  - Spectroscopy
- ***Mathematical background:***
  - Maxwell equations
  - Floquet-Bloch theory
  - Coupled-mode equations

Resonances

- cubic photonic crystal
- light propagation (Maxwell equations)

$$\nabla^2 \mathbf{E} - \frac{n^2}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \nabla\left(\nabla \cdot \mathbf{E}\right), \qquad \nabla \cdot \left(n^2 \mathbf{E}\right) = 0,$$

- $n = n(\mathbf{x})$ is the periodic refractive index:

$$n(\mathbf{x}) = n_0 \sum_{\mathbf{G}} \alpha_{\mathbf{G}} e^{i\mathbf{G}\mathbf{x}}$$

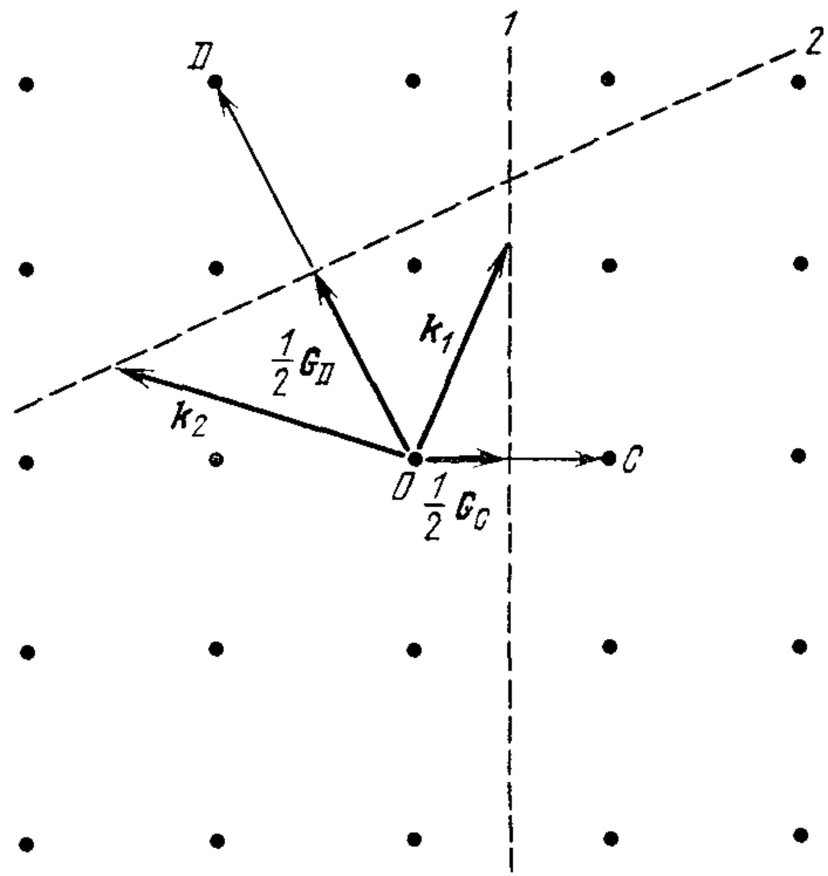
- Bloch waves:

$$\mathbf{E}(\mathbf{x}, t) = \mathbf{\Psi}(\mathbf{x}) e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)},$$

  where $\mathbf{\Psi}(\mathbf{x} + \mathbf{x}_0) = \mathbf{\Psi}(\mathbf{x})$ is the periodic envelope, $\mathbf{k} = (k_x, k_y, k_z)$ is the wave vector, and $\omega = \omega(\mathbf{k})$ is the wave frequency.

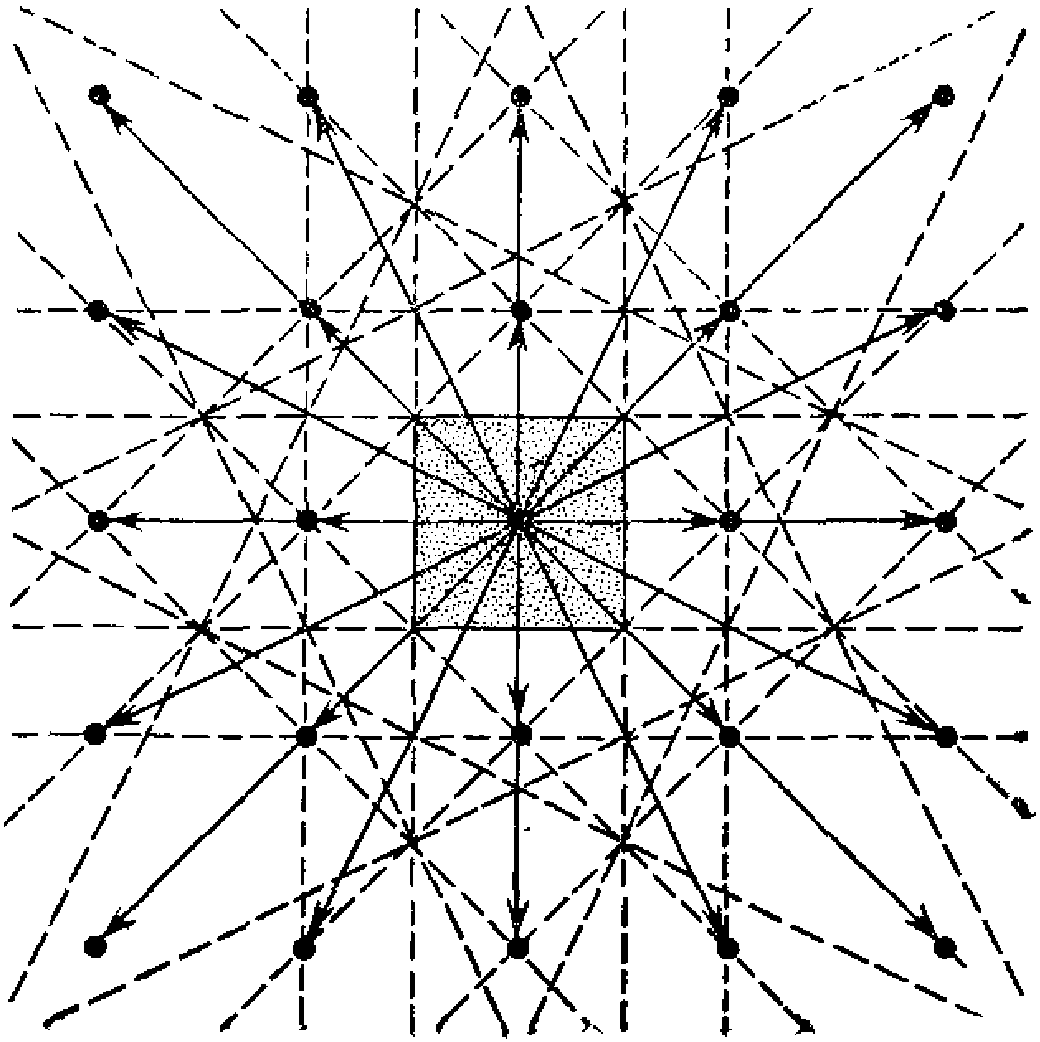
Resonances

$$\left.\begin{array}{l}\mathbf{k}' = \mathbf{k} + \mathbf{G} \\ |\mathbf{k}'| = |\mathbf{k}|\end{array}\right\} \Leftrightarrow \left.\begin{array}{l}\Delta\mathbf{k} = \mathbf{G} \\ (\mathbf{k}+\mathbf{G})^2 = k^2\end{array}\right\} \Rightarrow \begin{array}{l}2\mathbf{k}\cdot\mathbf{G} + G^2 = 0 \\ \mathbf{k}\cdot(\frac{1}{2}\mathbf{G}) = (\frac{1}{2}G)^2\end{array}$$



Brillouin construction

Resonances



Brillouin construction

Coupled-mode equations

Perturbation series expansions:

$$\begin{aligned} n(\mathbf{x}) &= n_0 + \epsilon n_1(\mathbf{x}) \\ \mathbf{E}(\mathbf{x}, t) &= \mathbf{E}_0(\mathbf{x}, t) + \epsilon \mathbf{E}_1(\mathbf{x}, t) + \mathrm{O}(\epsilon^2). \end{aligned}$$

Modulated resonant waves:

$$\mathbf{E}_0(\mathbf{x}, t) = \sum_{j=1}^{N} A_j(\mathbf{X}, T)\mathbf{e}_{\mathbf{k}_j} e^{i(\mathbf{k}_j \mathbf{x} - \omega t)}, \qquad \mathbf{X} = \frac{\epsilon \mathbf{x}}{k}, \quad T = \frac{\epsilon t}{\omega},$$

Inhomogeneous equation with resonances:

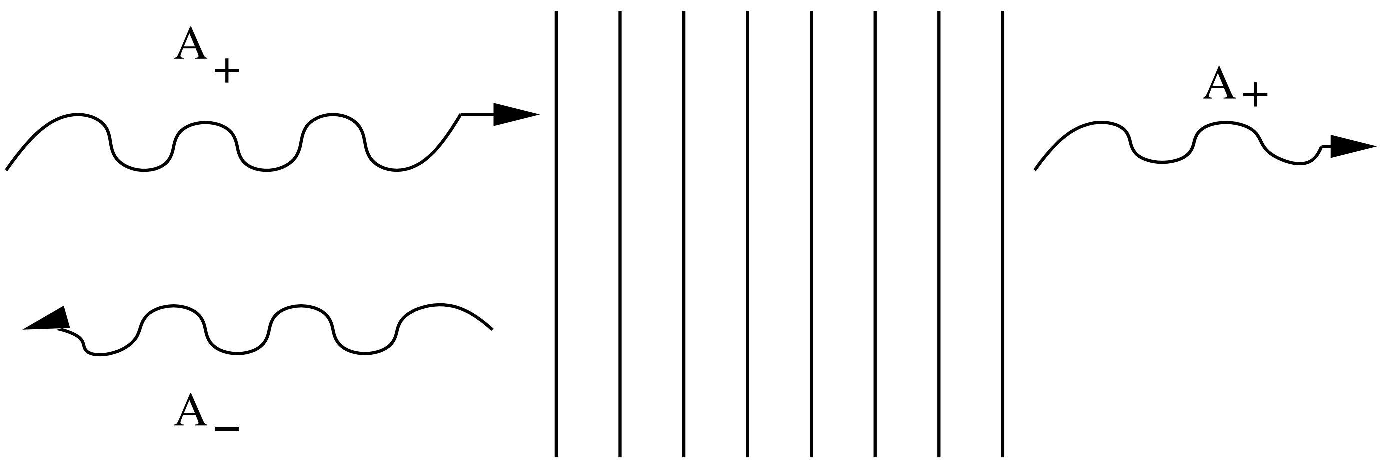
$$\nabla^2 \mathbf{E}_1 - \frac{n_0^2}{c^2}\frac{\partial^2 \mathbf{E}_1}{\partial t^2} = \mathbf{F}(\mathbf{E}_0),$$

Removal of resonant terms:

$$i\left(\frac{\partial A_j}{\partial T} + \left(\frac{\mathbf{k}_j}{k} \cdot \nabla_X\right) A_j\right) + \sum_{k \neq j} a_{j,k} A_k = 0, \qquad j = 1, ..., N,$$

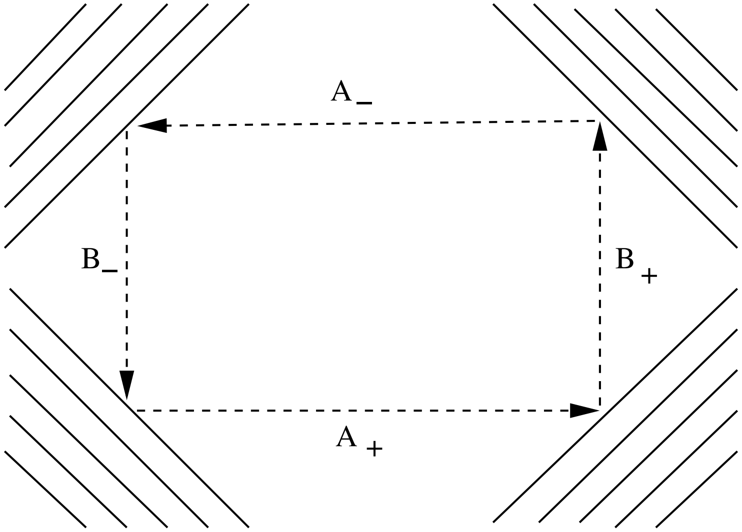
CME: Two waves

$$i\left(\frac{\partial A_+}{\partial T}+\frac{\partial A_+}{\partial Z}\right)+\alpha A_- = 0,$$
$$i\left(\frac{\partial A_-}{\partial T}-\frac{\partial A_-}{\partial Z}\right)+\alpha A_+ = 0,$$



CME: Four waves

$$i\left(\frac{\partial A_+}{\partial T}+\frac{\partial A_+}{\partial X}\right)+\alpha A_- +\beta\left(B_+ + B_-\right) = 0,$$
$$i\left(\frac{\partial A_-}{\partial T}-\frac{\partial A_-}{\partial X}\right)+\alpha A_+ +\beta\left(B_+ + B_-\right) = 0,$$
$$i\left(\frac{\partial B_+}{\partial T}+\frac{\partial B_+}{\partial Y}\right)+\beta\left(A_+ + A_-\right)+\alpha B_- = 0,$$
$$i\left(\frac{\partial B_-}{\partial T}-\frac{\partial B_-}{\partial Y}\right)+\beta\left(A_+ + A_-\right)+\alpha B_+ = 0,$$



Analysis

- **Stationary transmission:** $A_j(\mathbf{X}, T) = A_j(\mathbf{X})e^{-i\Omega T}$

$$i\left(\frac{\mathbf{k}_j}{k}\cdot\nabla_X\right)A_j + \Omega A_j + \sum_{k\neq j} a_{j,k}A_k = 0, \qquad j = 1, ..., N,$$

- **Existence and uniqueness of solutions for N waves:**
  - linear case
  - nonlinear case
- **Example: four counter-propagating waves**

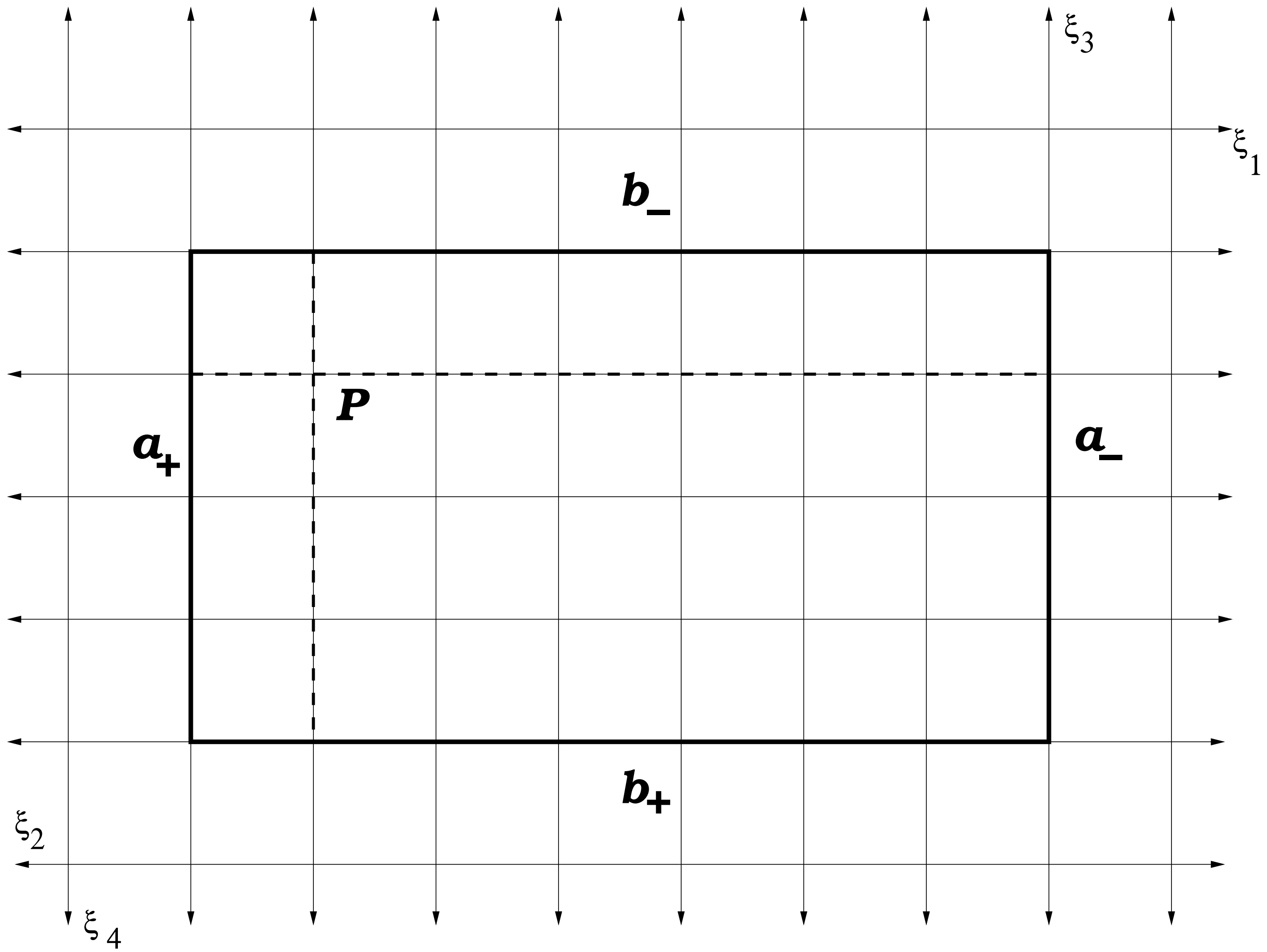
Analysis: Four waves

$$
\begin{aligned}
i\frac{\partial a_+}{\partial x} + \Omega a_+ + \alpha a_- + \beta\left(b_+ + b_-\right) &= 0,\\
-i\frac{\partial a_-}{\partial x} + \alpha a_+ + \Omega a_- + \beta\left(b_+ + b_-\right) &= 0,\\
i\frac{\partial b_+}{\partial y} + \beta\left(a_+ + a_-\right) + \Omega b_+ + \alpha b_- &= 0,\\
-i\frac{\partial b_-}{\partial y} + \beta\left(a_+ + a_-\right) + \alpha b_+ + \Omega b_- &= 0.
\end{aligned}
$$

Boundary-value problem on rectangle:

$$
\mathcal{D} = \{(x, y):\; 0 \le x \le L,\; 0 \le y \le H\},
$$

Analysis: Four waves



Four counter-propagating waves on the plane. Rectangle domain.

Analysis: Four waves

Theorem:

There exists a unique solution of the boundary-value problem.

Idea of Proof:

- **Let A be a continuous map of complete metric space R into itself such that A^n is a contraction; then $Au = u$ has a unique solution.**
- **The space R of continuous vector functions $\mathbf{v}(x, y)$ on the closed rectangle with the norm**

$$\rho(\mathbf{v}_1, \mathbf{v}_2) = \max_{x,y,i} |v_1^i(x, y) - v_2^i(x, y)|$$

is complete.

Analysis: Four waves

Linear case:

- **We transform the system to the integral form and consider iterations:**

$$v^1_{n+1}(x,y) = a_+(0,y) + \int_0^x (\Omega v^1_n + \alpha v^2_n + \beta(v^3_n + v^4_n))\,dx$$
$$v^2_{n+1}(x,y) = a_-(L,y) + \int_L^x (\alpha v^1_n + \Omega v^2_n + \beta(v^3_n + v^4_n))\,dx$$
$$v^3_{n+1}(x,y) = a_+(x,0) + \int_0^y (\beta(v^1_n + v^2_n) + \Omega v^3_n + \alpha v^4_n)\,dy$$
$$v^4_{n+1}(x,y) = b_-(x,H) + \int_H^y (\beta(v^1_n + v^2_n) + \alpha v^3_n + \Omega v^4_n)\,dy$$

  or symbolically

$$\mathbf{v}_{n+1} = A\mathbf{v}_n, \qquad \text{where } A \text{ is the integral operator}$$

- **We show that A^N is a contraction:**

Analysis: Four waves

$$|A\mathbf{v}_1 - A\mathbf{v}_2| \le \begin{pmatrix} x \\ L-x \\ y \\ H-y \end{pmatrix} M\|\mathbf{v}_1 - \mathbf{v}_2\|, \qquad M = |\Omega| + |\alpha| + |2\beta|$$

$$|A^n\mathbf{v}_1 - A^n\mathbf{v}_2| \le \begin{pmatrix} \frac{x^n}{n!} \\ \frac{(L-x)^n}{n!} \\ \frac{y^n}{n!} \\ \frac{(H-y)^n}{n!} \end{pmatrix} M^n\|\mathbf{v}_1 - \mathbf{v}_2\|$$

For any value of M, there exists a number N such that

$$\|A^N\mathbf{v}_1 - A^N\mathbf{v}_2\| \le \theta\|\mathbf{v}_1 - \mathbf{v}_2\|, \qquad \theta < 1$$

Analysis: Four waves

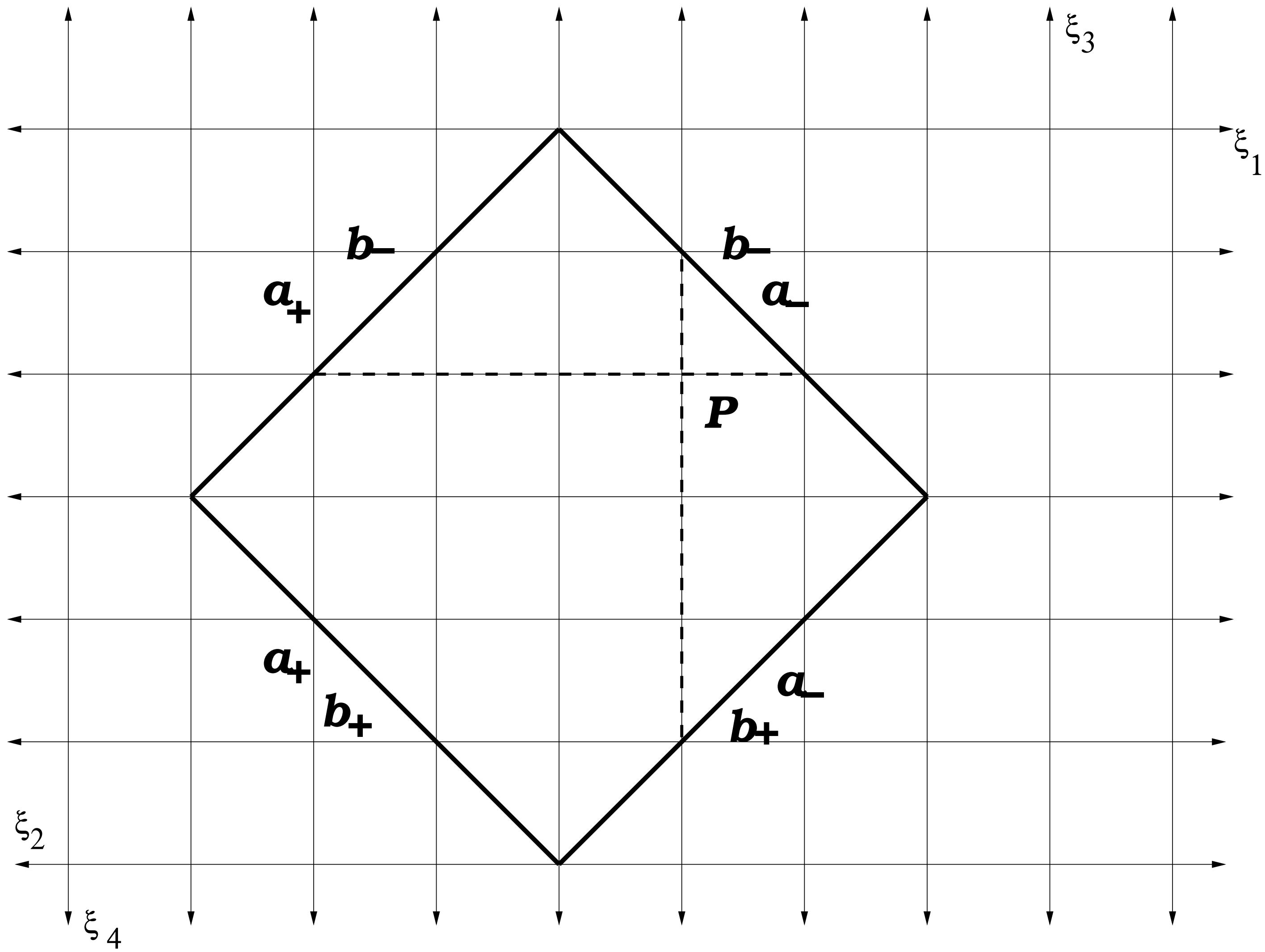
Non-linear case:

$$
\begin{aligned}
i\frac{\partial u^1}{\partial x} + F^1(x, y, \mathbf{u}) &= 0, \\
-i\frac{\partial u^2}{\partial x} + F^2(x, y, \mathbf{u}) &= 0, \\
i\frac{\partial u^3}{\partial y} + F^3(x, y, \mathbf{u}) &= 0, \\
-i\frac{\partial u^4}{\partial y} + F^4(x, y, \mathbf{u}) &= 0,
\end{aligned}
$$

with $\mathbf{F}(x, y, \mathbf{u})$ continuous and Lipschitz:

$$\|\mathbf{F}(x, y; \mathbf{u}_1) - \mathbf{F}(x, y; \mathbf{u}_2)\| \leq M\|\mathbf{u}_1 - \mathbf{u}_2\|$$

Analysis: Four waves



Four counter-propagating waves on the plane. Modified boundary value problem.

Analysis: bi-symplecticity

Denote:

$$h = \Omega(a_+\overline{a}_+ + a_-\overline{a}_- + b_+\overline{b}_+ + b_-\overline{b}_-) +$$
$$\alpha(\overline{a}_+a_- + a_+\overline{a}_-) + \alpha(\overline{b}_+b_- + b_+\overline{b}_-) +$$
$$\beta((b_+ + b_-)(\overline{a}_+ + \overline{a}_-) + (\overline{b}_+ + \overline{b}_-)(a_+ + a_-))$$

The system for four counter-propagating waves becomes:

$$\left.\begin{array}{l} \frac{\partial}{\partial x}a_+ = i\frac{\partial h}{\partial \overline{a}_+} \\ \frac{\partial}{\partial x}a_- = -i\frac{\partial h}{\partial \overline{a}_-} \end{array}\right\}, \qquad \left.\begin{array}{l} \frac{\partial}{\partial y}b_+ = i\frac{\partial h}{\partial \overline{b}_+} \\ \frac{\partial}{\partial y}b_- = -i\frac{\partial h}{\partial \overline{b}_-} \end{array}\right\}$$

Gauge symmetry $(a_+, a_-, b_+, b_-) \mapsto e^{i\phi}(a_+, a_-, b_+, b_-) \Rightarrow$ **conservation of flux**

$$\frac{\partial}{\partial x}\left(|a_+|^2 - |a_-|^2\right) + \frac{\partial}{\partial y}\left(|b_+|^2 - |b_-|^2\right) = 0.$$

Explicit solutions: 4 waves

Four-wave transmission system:

$$\begin{aligned} i\frac{\partial a_+}{\partial x} + \alpha a_- + \beta\left(b_+ + b_-\right) &= 0, \\ -i\frac{\partial a_-}{\partial x} + \alpha a_+ + \beta\left(b_+ + b_-\right) &= 0, \\ i\frac{\partial b_+}{\partial y} + \beta\left(a_+ + a_-\right) + \alpha b_- &= 0, \\ -i\frac{\partial b_-}{\partial y} + \beta\left(a_+ + a_-\right) + \alpha b_+ &= 0. \end{aligned}$$

On the rectangle $\mathcal{D} = \{(x, y) : \; 0 \leq x \leq L, \; 0 \leq y \leq H\}$,
Boundary conditions:

$$a_+(0, y) = \alpha_+(y), \quad a_-(L, y) = 0, \quad b_+(x, 0) = 0, \quad b_-(x, H) = 0$$

Explicit solutions: 4 waves

Separation of variables:

$$a_+(x,y) = u_+(x)w_a(y), \qquad a_-(x,y) = u_-(x)w_a(y)$$
$$b_+(x,y) = w_b(x)v_+(y), \qquad b_-(x,y) = w_b(x)v_-(y),$$

where

$$v_+(y) + v_-(y) = \mu w_a(y), \qquad u_+(x) + u_-(x) = -\lambda w_b(x),$$

$$\begin{pmatrix} i\partial_x & \alpha \\ \alpha & -i\partial_x \end{pmatrix} \begin{pmatrix} u_+ \\ u_- \end{pmatrix} = \beta\Gamma^{-1} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u_+ \\ u_- \end{pmatrix}$$
$$\begin{pmatrix} i\partial_y & \alpha \\ \alpha & -i\partial_y \end{pmatrix} \begin{pmatrix} v_+ \\ v_- \end{pmatrix} = \beta\Gamma \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} v_+ \\ v_- \end{pmatrix}, \qquad \Gamma = \lambda/\mu.$$

The boundary conditions for ODE systems:

$$u_+(0) = 1, \quad u_-(L) = 0,$$

and

$$v_+(0) = v_-(H) = 0.$$

Explicit solutions: 4 waves

- **The homogeneous problem for $(v_+, v_-)^T$ defines the spectrum of Γ:**

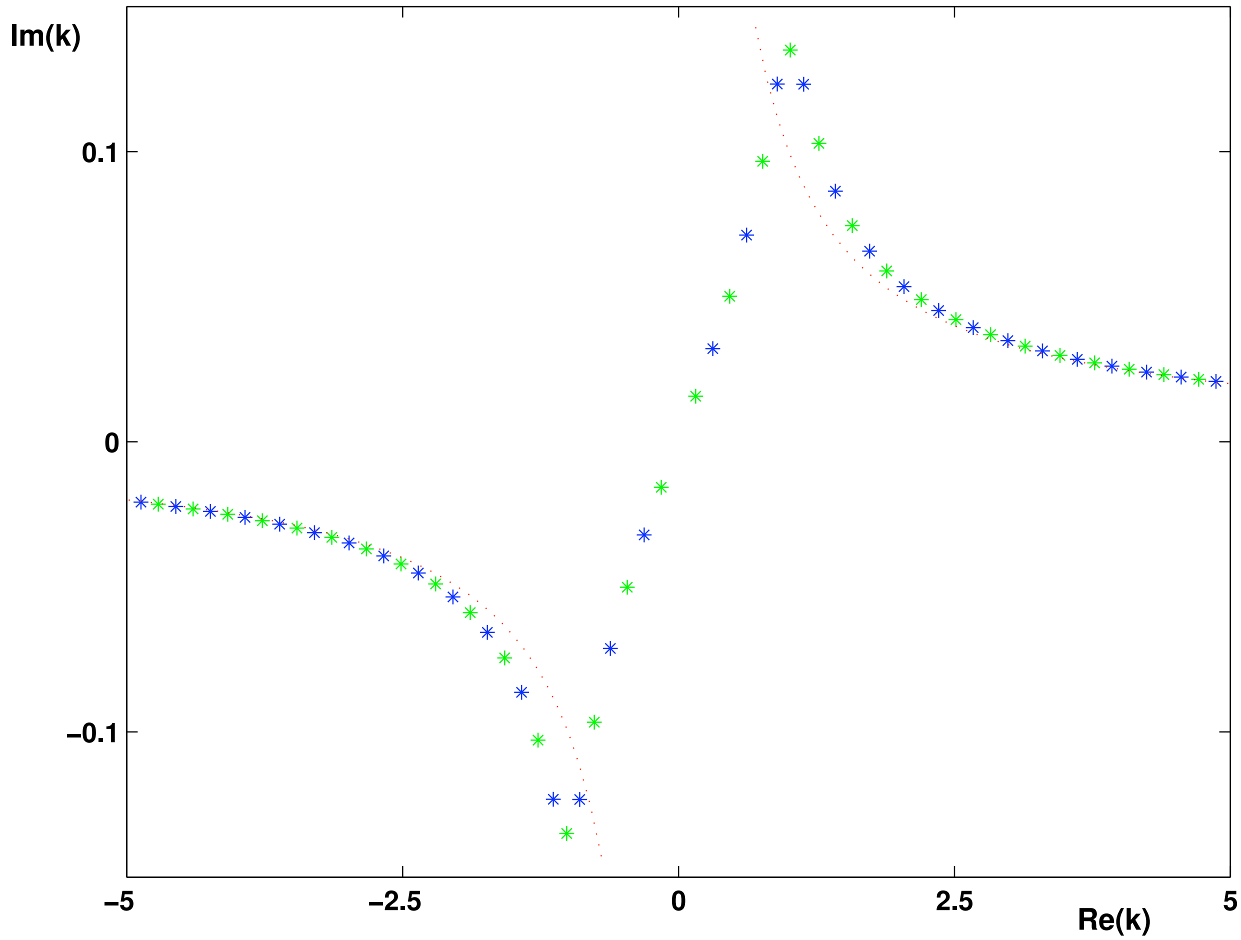
$$\Gamma = \frac{\alpha^2 + k^2}{2\alpha\beta},$$

where

$$\left(\frac{k-\alpha}{k+\alpha}\right)^2 e^{-2ikH} = 1$$

- **The inhomogeneous problem for $(u_+, u_-)^T$ defines a unique particular solution**

Explicit solutions: 4 waves



Roots of the characteristic equation $\left(\frac{k-\alpha}{k+\alpha}\right)^2 e^{-2ikH} = 1$

Explicit solutions: 4 waves

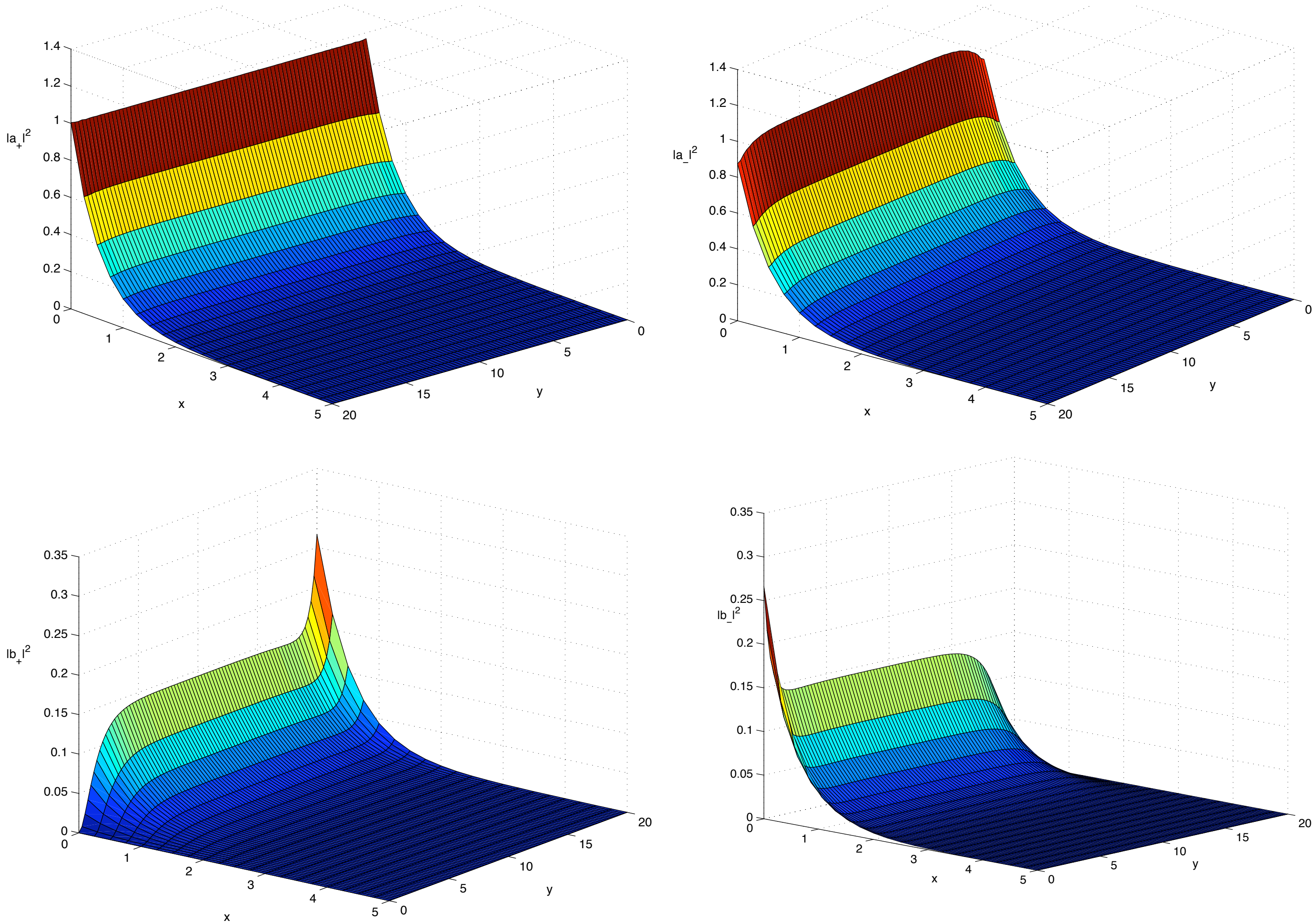
The set of eigenfunctions $v(y) = v_+(y) + v_-(y)$ **is orthogonal and complete, such that:**

$$\alpha_+(y) = \sum_{\text{all } k_j \in \mathcal{R}} c_j v_j(y), \qquad c_j = \int_0^H \alpha_+(y) v_j(y) dy,$$

Solution:

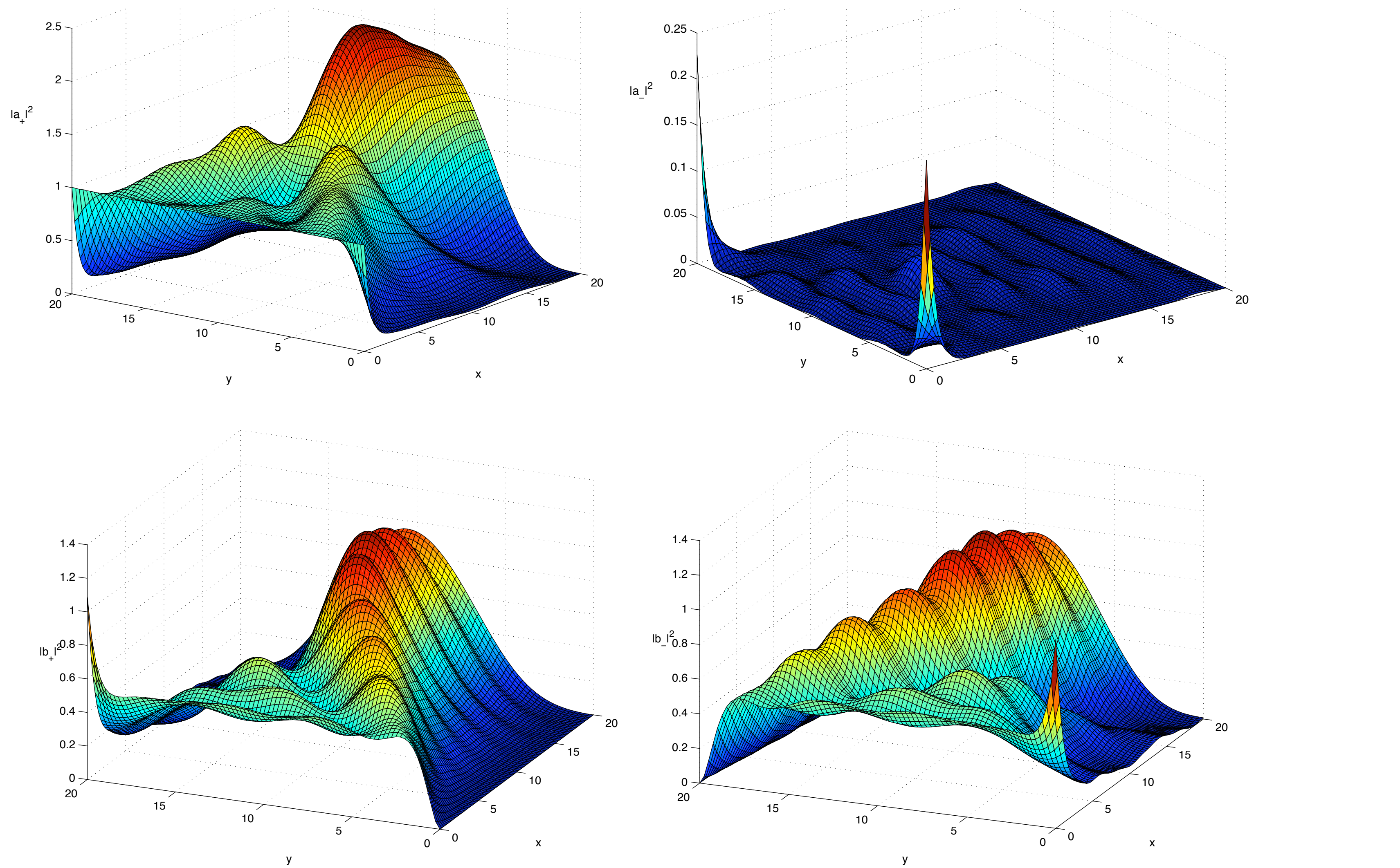
$$\begin{aligned}
a_+(x,y) &= \sum_{\text{all } k_j \in \mathcal{R}} c_j \frac{u_{+j}(x)}{u_{+j}(0)} \left(v_{+j}(y) + v_{-j}(y)\right), \\
a_-(x,y) &= \sum_{\text{all } k_j \in \mathcal{R}} c_j \frac{u_{-j}(x)}{u_{+j}(0)} \left(v_{+j}(y) + v_{-j}(y)\right), \\
b_+(x,y) &= -\sum_{\text{all } k_j \in \mathcal{R}} c_j \frac{u_{+j}(x) + u_{-j}(x)}{\Gamma_j u_{+j}(0)} v_{+j}(y), \\
b_-(x,y) &= -\sum_{\text{all } k_j \in \mathcal{R}} c_j \frac{u_{+j}(x) + u_{-j}(x)}{\Gamma_j u_{+j}(0)} v_{-j}(y).
\end{aligned}$$

Explicit solutions: 4 waves



Solution surfaces $|a_{\pm}|^2(x,y)$ and $|b_{\pm}|^2(x,y)$ for $\alpha = 1$, $\beta = 0.25$, $L = H = 20$, and $\alpha_+ = 1$.

Explicit solutions: 4 waves



Solution surfaces $|a_\pm|^2(x,y)$ and $|b_\pm|^2(x,y)$ for $\alpha = 1$, $\beta = 0.75$, $L = H = 20$, and $\alpha_+ = 1$.

Summary

Results:

- **The existence and uniqueness theorem for N waves**
- **Analytical solution for four counter-propagating waves**

Open problems:

- **Non-stationary transmission**
- **Multi-symplectic structure of coupled-mode equations**
- **Feynman diagram technique**
- **Numerics**

References

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The End

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